APPLICATIONS OF MATRIX METHOD AND GROUP THEORY TO SYMMETRICAL STRUCTURAL SYSTEMS

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CERTIFICATE

This is to certify that the thesis titled "Applications of Matrix Method and Group Theory to Symmetrical Structural Systems" is a bonafide work done under my supervision and has not been submitted elsewhere for the award of any degree.

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ABSTRACT

Symmetry of structural systems are defined in terms of symmetry elements and symmetry operations. It is found that the stiffness matrix of any system may exhibit certain symmetries directly related to the symmetrie of the system. These symmetries of the stiffness matrix are found to be their commuting property with a set of matrices knowy as the symmetry transformation matrices. When the stiffness matrix is similarity transformed by the matrix formed from the eigen vectors of the symmetry transformation matrices, it gets block diagonalised. It is also shown that load systems have nothing to do with the symmetry of structural systems.

The procedure is applied to reflection symmetric, double reflection symmetric, cyclically symmetric (and their combinations) structural systems. However, some inherent difficulties and inadequacies are found to be present in such a procedure, e.g. (i) the procedure can not be applied if nodes are comming on the symmetry planes or symmetry axes, (ii) the procedure can not yield any information about the effects of increasing or decreasing the symmetry from the out set, (iii) the choice of the co-ordinate system and the numbering of degrees of freedom should be of particular types.

However it is found that the symmetry elements of structural systems form groups (called symmetry group of the system) and therefore all the mathematics of the group theory is applicable for dealing with the symmetry of the structural systems. It is

found that the stiffness matrix of any structural system gets block diagonalised in a co-ordinate system which is generated by the projection operators of the symmetry group of the structural system. It is seen that all the difficulties encountered in the classical matrix method are resolved i.e. one can apply symmetry arguments to simplify the problems by using group theory even if the nodes are comming on the symmetry axes or the symmetry planes. The numberings of the degrees of freedom and the initial choice of the nodal co-ordinate systems are in no way restricted. The effects of increasing or decreasing the symmetry elements are also seen from the out set.

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NOMENCLATURE

(a) CAPITAL LETTERS WITHOUT SUFFICES

A, B, C, D	:	Sub-matrices of stiffness matrix
E	:	Identity symmetry element
F	:	Force vectors
G	:	Groups, Force vectors
H	:	subgroups of groups
I	:	Identity matrix
K	:	Stiffness matrix
L	:	Linear operator
M	:	Mass matrix
N	:	Number of degrees of freedom
P	:	Force vectors
R	:	Representations
S	:	Member stiffness matrices, Similarity transformation matrices.
T	:	Transformation matrices
U	:	Deformation potential energy
V	:	General Vectors
X	:	Displacement vectors
Y	:	Displacement vectors

(b) CAPITAL LETTERS WITH SUFFICES:

A_i	:	Transformation matrices, Submatrices of stiffness matrices
B_i	:	Sub-matrices of stiffness matrices
C_i	:	Sub-matrices of stiffness matrices

C_n	:	Axis of symmetry, Symmetry Element, Cyclic Group of order n.
C_{nv}	:	The groups containing C_n group and n-planes of symmetry passing through C_n.
C_{nh}	:	The groups containing C_n group and one plane of symmetry perpendicular to C_n.
$D(g_i)$	:	The representation matrices of any group
D_n	:	The group containing n two-fold axis along with C_n group.
D_{nh}	:	The group containing a reflection plane perpendicular to C_n along with C_n group.
K_l	:	The l-th class of a group, class sum of the l-th class
K_{ij}	:	The submatrices of stiffness matrices.
$P_{ij}^{(k)}$	:	Projection operator for the k-th irreducible representation with ij element of the representation matrices.
R_m	:	The m-th irreducible representation of any group.
S_n	:	The symmetry element $S_n = \sigma_h \; C_n$, group containing S_n and its powers.

(c) SMALL LETTERS

a, b, c, d	:	Any variable
e	:	Basis, member deformation vector
e_i	:	Basis vectors
g_i	:	ith element of any group

h_i	:	ith element of any subgroup of a group
i	:	Inversion symmetry element, any running suffix
k	:	Character of a group element
r	:	Number of classes in any group; any running suffix
r_i	:	Number of elements in i-th class.

(d) GREEK LETTERS

σ	:	Reflection plane, and the corresponding symmetry plane operation
σ_m	:	m-th reflection plane and the corresponding symmetry operation
σ_v^m	:	m-th reflection plane passing through the principal axis and the corresponding symmetry operation
σ_h	:	Reflection plane perpendicular to the principal axis and the corresponding symmetry operation
α	:	Angle, Roots of a polynomial
θ_m	:	m-th root of unity
ψ	:	A general eigen function.

CHAPTER 1

INTRODUCTION AND LITERATURE REVIEW

The word symmetry has been derived from the Greek word symmetria (1) which literally means harmony, balance in proportions of parts to the whole, repetitions of similar objects etc.etc. This sort of meaning of the word symmetry is vague and is not all that for what it is used in daily life. It becomes essential to understand the context in which it is referred to e.g. one says symmetry of head and tail of a coin determines the probability of occurrence of head or tail to be $\frac{1}{2}$ if coin is tossed, the equation of equilibrium for in-plane forces for shells or plates in one direction can be immediately written from that of the second direction from symmetry. From symmetry the laws of mechanics should hold good at Mars etc. It is now very clear that defining symmetry in particular terms is not at all adequate and a generalised definition is needed. This job can be accomplished by modern mathematical language which talks with no reference to particular objects or

phenomena as has been very clearly discussed by Sawyer (2). This will be the language followed in this thesis.

Looking back in to history, one can see that the development of the concept of symmetry starts on two grounds, one, on philosophical grounds and the other from aesthetical grounds.

Indian Upanisdic philosophy talked of symmetry of Universe and Brahman by saying that Brahman has transformed itself to create universe while weastern philosophers talked of God putting life in Adam from His right hand through Adam's left hand. This latter view shows the concept of mirror symmetry between God and Adam but at the same time distinguishes left from right. The scientific thinking till recently has sided the former. (Recently Lee has demonstrated the distinction between left and right the so called "Non-conservation of parity"). The important philosophical findings of Aristotle is also worthy of mentioning here. He argued that universe must be spherical because God likes symmetry and the most symmetrical shape is sphere. Needless to say the laws of mechanics have always been discovered from the concept of symmetry. The symmetry of "here" and "there",

the symmetry of "now" and "then" constitutes the laws of classical mechanics including the theory of relativity of Einstien. The symmetry of particles and waves has given rise to quantum mechanics (3, 4, 5). These symmetries were explicitly exploited by Lagrange, Hamilton Jacobi and Neother. They formulated mechanics as invariance principle, which yielded the conservation laws of mechanics as a corollary due to various symmetries of nature (6). These points have been beautifully discussed by Lanzocs (7).

Egyptians, Babylonians and Chinese used symmetry in the architectural designs of buildings, windows, paintings etc. An extensive account of this aspect has been given by Hermann Weyle in his monograph on Symmetry (8). Plato and Aristotle maintained that symmetry is both necessary as well as sufficient for beauty. In this context Plato's traingle is very famous. He says, "Now, the one which we maintain to be the most beautiful of all many traingles is that of which the double forms the third traingle which is equilateral". He further says that measure and porportion always pass into beauty and excellence. Almost a similar view was also

held by Aristotle. However latter mathematecians and philosophers like Descartes, Leibnitz, Euler, Helmholtz, Syleœester etc. argued that symmetry is only a necessary condition for beauty. An exhaustive study has been done by Birkhoff (9) who again emphesizes the symmetry as a necessary condition for beauty. He gives a mathematical formula for the aesthetic measure of an object belonging to a class in which the comparision of aesthetic value is possible. His formula states that aesthetic value of an object is proportional to its order and inversely proportional to its complexity. Hamlin gave his first law of archetecture which emphesized the presence of mirror symmetry of a building (1) and most probabily this is the reason most of the structures are having mirror symmetry. From all this it can be seen that earlier people have meant two different kinds of symmetries, physical and geometrical and have used these mostly qualitatively or in general terms only. People of recent times have made use of this concept in more fascinating way. As an example consider the following problem of calculus of variation: Given an equilateral triangle (Fig.1·1),it is required to divide this by a curve joining one point each from one of the two opposite sides such

that area of the triangle is divided and at the same time the curve has a minimum length.

This problem is an extremely difficult problem of calculus of variation. However, if the concept of symmetry is applied, the problem becomes extremely simple. Let us go on rotating the triangle about one of the two sides AC or BC 5-times till a closed hexagon is formed as shown in Fig. 2. The arc of Fig. 1 form now a closed curve inside the regular hexagon. Now, since one knows that for a given area, the curve which has the least perimeter is circle. Hence PQ must be an arc of a circle and so the problem is solved, because if a is side of the triangle then $CP = CQ = a\left(\frac{3\sqrt{3}}{4\pi}\right)^{\frac{1}{2}}$. Similar arguments can be applied for isocles triangles and for many other geometrical figures. Similar examples can be found in a recent book by Saaty (10).

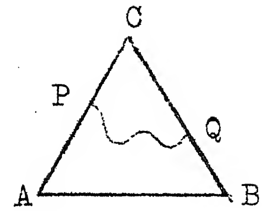


Fig. 1.1

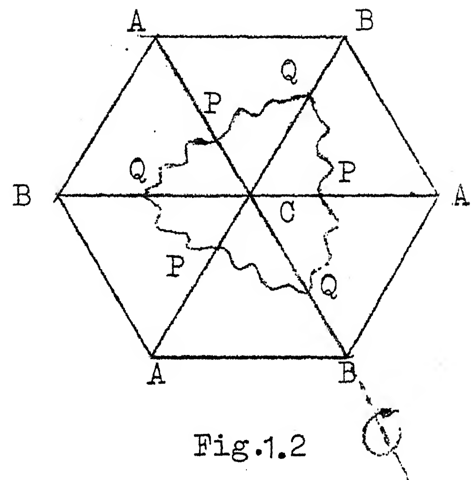


Fig.1.2

Coming to the structural mechanics one finds that Maxwell was one of the poineer interested in this

aspect of structural mechanics. His reciprocaty theorem is known to every one. This symmetry is due to the inherent symmetry of assumed linear laws and has no connection to the geometric symmetries inflicted by designers. In this regard it is worth mentioning that a similar reciprocaty theorem was derived by Green for electrostatic conductors much earlier to Maxwell's theorem for elastostatic (11, 12 and 13). What Maxwell had to do most probabily was to change the terminology. In fact electrostatics and elastostatics are mathematically one and the same if one describes the latter by a set of generalised co-ordinates which can be seen from the following.

Electrostatics	Elastostatics
Set of conductors	Set of nodal degrees of freedom
Charges $(Q_1, Q_2, \ldots Q_n)$	Displacements on the nodal points $(X_1, X_2, --- X_n)$
Potentials $(V_1, V_2, \ldots V_n)$	Forces on the nodal points $(F_1, F_2, --- F_n)$

$$\begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_n \end{bmatrix} = \begin{bmatrix} C \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix} \qquad \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{bmatrix} = \begin{bmatrix} K \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}$$

$$W = \tfrac{1}{2} Q^T S Q = \tfrac{1}{2} V^T C V \qquad W = \tfrac{1}{2} X^T K X = \tfrac{1}{2} F^T A F$$

$$C = C^T \quad S = S^T = C^{-1} = (C^{-1})^T \quad K = K^T \quad A = A^T = K^{-1} = (K^{-1})^T$$

Due to Green (1828) Maxwell (1869)

Hence the reciprocal theorem should be called Green-Maxwell-Beti reciprocal theorem or just Green's theorem. There is another theorem due to Maxwell which reads as

$$V_t \sigma_t - V_c \sigma_c = \sum_{i=1}^{n} \vec{F}_i \vec{r}_i \qquad (14-16)$$

where V_t, V_c, σ_t, σ_c are the volumes and stresses in tension and compression members respectively and $\vec{F}_i$, $\vec{r}_i$ are the nodal forces and position vectors w.r.t. an arbitrary origin respectively. In some cases

it can be seen that this theorem implies that symmetry is an "optimality" in structural design. That symmetry is "Optimality" has been discussed by Kiefer (17) in connection with decision theory depending upon the nature's move. Intuitively one always first thinks of structures with certain symmetry which turn out to be optimal in some cases but attention has not been given to the role of symmetry in optimization.

Another area of structural mechanics where symmetry has been utilised is the "Buckling of frames". The workers of this field however have exclusively associated the symmetries of Loads with that of structures without clearifying the reasons of doing so. In this area people have followed two approaches, one is that of guessing the mode shapes in tune with the symmetry of structural systems and load systems (18, 19) and the other is that of hypothetically dividing the whole structural system in to symmetrical components and then to follow the various algebraic routines available (20,21,22,23). The former approach except for trivial cases of mirror symmetry where one guesses the so called symmetric and antisymmetric mode shapes, seems to be cumbersome and unjustified while the latter creates confusion with

regard to the diversity of available algebraic methods and unsound physical reasoning. Also, these methods lack in explaining the role played by load symmetry. One does not understand as to which one of the structural symmetry or the load symmetry is more important.

The literature is full of the examples with mirror symmetry but more complicated symmetrical structures have been dealt only by few. The note worthy is the work of Stavaraki (18) who applied the intitutive method of guessing the modesahpes for hexagonal space frame. Renton (20,21) has discussed various types of symmetries and applied the theorem of Burnside to factorise the determinantal equation determining the critical loads. His approach is confusing due to the use of various results without any explaination. As discussed above, the approach presumes the similar symmetry for load system also without explaining why it does so. Also, the procedure by its inherent inadequate methodology fails to deal with analysis and vibration problems along with buckling at a stretch.

Hussey (22) divides the cyclically symmetric system into various symmetric components what he calls

the subframes and then applies arguments of symmetry to simplify the structure of total stiffness matrix. After that, he applies what is known as "finite Fourier transform" without explaining as to what makes him to do that. Once again, as pointed out earlier, he mixes up the force system and structural system in arguing the symmetry.

Salem (23) has discussed the mode-shapes for buckling of mirror symmetric frames with mirror symmetric loads. He quotes two statements determining the mode shapes and critical loads, without giving any proof. He justifies his statements by giving various examples. His first statement which is universal follows immediately as a corollary of the present work discussed in 3rd Chapter. His second statement is a conditional one and depends upon the inner structure of the problems and does not follow from the mere symmetry.

Another field is that of free vibrations of the structural systems. To author's knowledge except for the trivial case of mirror symmetry no one has applied the symmetry arguments in this area. Only physicists have made use of the symmetries of molecules in predicting

their mode shapes. Various excellent books are available on this topic (24-30).

Yang (31) has applied the arguments of symmetry to predict the mode-shapes of non-linear vibrations of symmetric systems. He uses the concept of phase space and divides the whole phase space into invariant sub-spaces. The concept is unnecessarily complicated and a partial amount of informations can be obtained by the method of III Chapter of the present work. The Chapter V can yield all the results of Yang more excellently.

One of the important area in which symmetry has been understood to be the mirror symmetry only is structural analysis. People have handled the mirror symmetry by breaking the total problem in two sub-problems what they call to be the symmetric and anti-symmetric parts of the problem. It can be easily seen that such an approach is bound to fail if there are complicated symmetries e.g. two orthogonal planes of symmetry, cyclic symmetry etc. In this regard Broek's (32) two rules what he calls the rule for symmetry and the rule for antisymmetry are interesting to note:

Rule for Symmetry:

"A structure subject to a certain loading condition is said to be symmetrical about an axis of symmetry when, on being turned about this axis through 180°, the resulting structure and loading are identical.with the original.

Rule for Antisymmetry:

A structure subject to a certain load condition is said to be anti-symmetrical about an axis of antisymmetry when, on being turned about this axis of anti-symmetry through 180° and the sense of the loading reserved, the resulting structure and loading are identical with the original.

The load here once again is being associated with structural system while applying the symmetry arguments. This is not at all needed.

People (33) have not recognised the symmetry of the system like stair frame shown in Fig. 1.3

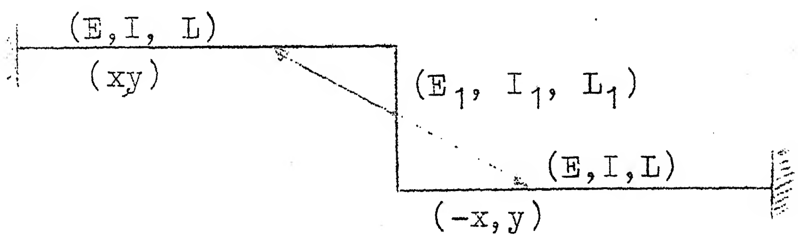


Fig. 1.3 Star frame with inversion symmetry

The excellent monograph of Parkes (16) also illustrates the use of symmetry for simplifying the mirror symmetric structural problem. Almost all books on structural analysis illustrate the use of mirror symmetry for simplifying the analysis problems by using the similar approach as that of Parkes (16). In a nutshell the meaning of symmetry in structural analysis has confined to mirror symmetry only for which people have found it convenient to split the problem in symmetric and anti-symmetric parts of the total problem.

In continuum mechanics however symmetry has taken profound attention and specially in non-linear continuum mechanics one needs many many symmetry arguments to simplify the constitutive relations. The work of Green and Adkins (34) is noteworthy. However there is a basic difference in symmetry of a continuum and that of a structural system. In a continuum one works with a combination of what are called point symmetries and translational symmetry. While in structural mechanics one has to deal with only point symmetry viz. in a continuum one concentrates just at one point of the continuum and hence has to consider the mechanical behaviour only in different direction at that point, while in structural mechanics

the whole of the structure has to be considered at a time and hence the corresponding symmetries have to be used. Note that the point symmetry does not mean that symmetry at a point but that symmetry in which at least a point remains unchanged under symmetry transformations. This aspect will be dealt in detail in the following Chapter.

Recently, the use of algebraic invariants to continuum mechanics has taken much attention (35). Invariant and symmetries are very closely connected.

The second chapter developes the language for describing symmetry and deals with the various kinds of possible structural symmetries with the help of what have been termed as symmetry operations and symmetry elements. It is shown that the symmetry of structural system has nothing to do with the symmetry of load systems. In order to be able to distinguish between the configurational symmetry and elemental symmetry a different definition for the structural system is evolved and from that it is shown that even without configurational symmetry, there can be structural symmetry so far as

its role to simplifications of problem is concerned. The structural symmetry can show up in total stiffness or flexibility matrix in the form of matrix symmetries. For this a broad meaning of matrix symmetry is given.

The third chapter starts with mirror symmetry and goes upto cyclic-reflection symmetry. Side by side the structural symmetry is cast into stiffness matrix and the resulting simplification is worked out. Illustrative examples are also given for each kind of symmetry. The chapter considers analysis (bending), stability, and vibration problems at a stretch. Finally the inadequacies of the method, which are present in almost all classical method of using the symmetry arguments are pointed out, and so a need for better algebra is felt.

Then in the fourth chapter "Group Theory" (the mathematics of symmetry) is developed from physical view point.

The fifth chapter then exclusively deals with the application of group theory to symmetric structural problem and various difficulties encountered in the Third Chapters are shown to get resolved by the use of this approach.

CHAPTER 2

THE STRUCTURAL SYMMETRY

2.1 SYMMETRY OPERATIONS AND SYMMETRY ELEMENTS:

To start with, consider the simplest mechanical system, the spring mass system shown in Fig. 2.1. Let the corner masses be named A, B, and C. Since the masses and springs are exactly similar, one can not physically distinguish between these masses. However, in order to describe any configuration of any mechanical system one needs a co-ordinate system. Thus in Fig. 2.1 there are three possible co-ordinate systems w.r.t. which one has exactly similar configurations, i.e. description of mechanical state w.r.t. any of the three co-ordinate system is exactly same except for the name which does not play any role in mechanics. Here if one looks carefully, observes that the co-ordinate system of (b) and (c) are obtained by rotating the co-ordinate system of (a) by 120° and 240° respectively. Thus, if instead of rotating the co-ordinate system, one rotates the mechanical system itself by 120°, 240° respectively in

opposite direction to that of earlier rotation, one gets three configurations shown in Fig. 2.2 with a single co-ordinate system. As discussed earlier, these configurations are indistinguishable; i.e. their descriptions w.r.t. the chosen co-ordinate system is one and the same so far as mechanical behavior is concerned. The rotations by 120^0, 240^0, therefore, do not change the system. Putting in another way, the cyclic interchanges of the masses A, B, and C do not change the system. Such interchanges which do not change the configuration of a mechanical system are called symmetry operations. One question arises here. Why only the cyclic interchanges of the masses do not change the configuration? Will any arbitrary interchange affect the mechanical system? The answer to these questions is negative. So the definition of symmetry operation is:

"A symmetry operation in a mechanical system is any operation whereby some or all identical parts of the system are moved about in such a way that the result is indistinguishable from the original mechanical system".

For the sake of completeness identity operation or no operation is also given as one of the symmetry operation. Hence every mechanical system is having at least one symmetry operation (the identity operation) under which the system results into itself.

From the definition of the symmetry operation it is seen that every operation is associated with a mathematical entity known as "mapping". These are special types of mappings and are named as "symmetry elements". Thus if a symmetry element is associated with a given mechanical system, then under the operation of this element (what has earlier been called as symmetry operation), the resulting configuration of system remains indistinguishable from the original configuration. These elements are basically the following:

1. Reflection across a plane, about a point etc.
2. Rotation about an axis
3. Translation

and a combination of all these.

e.g. a if a structure has two symmetry elements:

(i) Identity : E

(ii) A reflection plane : σ

then the structure is said to be mirror symmetric and if it has n-elements:

Identity, E

A rotation by $\frac{2\pi}{n}$ named C_n

A rotation by $2\frac{2\pi}{n}$ named C_n^2

. . . .

and A rotation by $(n-1)\ \frac{2\pi}{n}$ named C_n^{n-1}

Then the structure is said to be cyclically symmetric.

Referring to the Fig. 2.1 or 2.2 one can see that there are 6-elements of symmetry under which the system's configuration remains indistinguishable. These are following:

Rotation by 120° about an axis passing through the center of the system named C_3

Rotation by 240° about an axis passing through the center of the system named C_3^2.

Reflection about the plane perpendicular to the plane of the system and passing through "A" and the middle point of the line joining BC named σ_v^a.

Similar reflections passing through B and C and the middle points of the springs opposite to them, named respectively as σ_v^b and σ_v^c.

Thus for the above system one has the following set of symmetry elements:

$$E,\ C_3,\ C_3^2,\ \sigma_v^a,\ \sigma_v^b,\ \sigma_v^c$$

2.1.1 Notations and Terminology:

C_n : stands for following:

(i) symmetry element corresponding to a rotation by $\frac{2\pi}{n}$

(ii) Axis of rotation of order n.

C_n^m : stands for an element corresponding to a rotation by $m\,\frac{2\pi}{n}$

mC_n : stands for a set of symmetry elements with 'm' members, namely $C_n,\ C_n^2,\ C_n^3\ ---\ C_n^m$.

Note that a system may have more than one axis of symmetry of different or same order e.g. a sphere has an infinite number of axes of symmetry of infinite order, a tetrahedral frome shown in Fig. 2.3 has 4 axes of symmetry each of order 3, the frame shown in Fig. 2.16 has one three-fold axis and three two-fold axes of symmetry. In such cases, the axis with highest order is called as principal axis.

σ_v : stands for reflection across a plane (or the reflection plane) other than the one which is perpendicular to the principal axis.

σ_h : stands for reflection across that plane (or the reflection plane) which is perpendicular to the principal axis of symmetry.

σ : stands for reflection (or the reflection plane) when there is only one reflection plane and no axis of symmetry is considered.

σ_m : stands for m-th reflection or reflection plane when there is no reflection plane perpendicular to the axis of symmetry.

i : stands for inversion under which every point (x, y, z) is changed to (-x, -y, -z) and for a planner system it is meant that under this a point (x,y) is changed to (-x, -y).

Note that the planer inversion is nothing but a C_2.

The examples of inversion symmetric systems are frames shown in Figs. 1.3 and 2.4

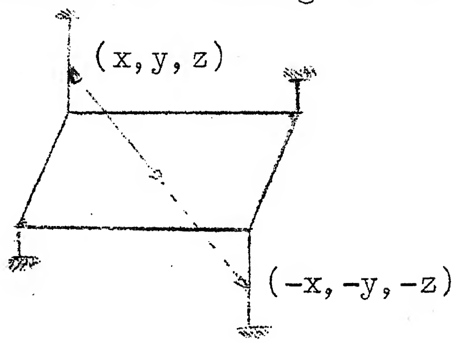


Fig. 2.4 Inversion symmetric frame

Consider now the frame shown in Fig. 2.16. Let us rotate the structure about C_3 by 120° and then reflect across σ_h. This operation may be written as $\sigma_h C_3$. The result of this operation is seen from the following scheme

$$\sigma_h C_3 \begin{pmatrix} A & B & C \\ A' & B' & C' \end{pmatrix} = \sigma_h \begin{pmatrix} C & A & B \\ C' & A' & B' \end{pmatrix} = \begin{pmatrix} C' & A' & B' \\ C & A & B \end{pmatrix}$$

Please refer to page no. 29

Where A, B, C and A', B', C' are nodes of structure and C_3 rotates by 120 in counter clockwise direction and then σ_h interchanges the lower and upper-nodes. It can be seen now that the resulting configuration is not obtainable from any of the symmetry elements given earlier. Thus $\sigma_h C_3$ is a new symmetry element.

These symmetry elements are called improper rotations, in the sense that they involve reflections along with rotations. The notation for these is given as below.

S_n^k : stands for a rotation by $\frac{2\pi k}{n}$ followed by a reflection across a plane perpendicular to C_n.

So far the translation has not been discussed. This symmetry element is usually absent from engineering structures because of their finite extent. However to an approximation one can assume this element to be present in structural system as has been done by Renton (20, 36). By introducing this basic element of symmetry one can generate many many new symmetry elements by multiplying this element with all other elements. Usually there are two types of translational symmetry elements defined as follows:

T : a simple translation where one part of the system is translated by one unit in a given direction, without any resulting change in configuration of the system. (See Fig. 2.5, 2.6 and 2.7).

T(t) : a magnified translation, where translation is followed by a magnification of what is being translated and also, the unit of translation is changing with the number of translations performed. This generates what is known as repetitive magnified symmetry (8) and is shown in (Fig. 2.8).

The translational symmetry is not discussed here any more for the simple reason that it requires either an infinite extent, or a cyclic structure. The former does not exist in engineering and the latter has already come in C_n etc. and so the basic symmetry elements are only

$$E, \quad \sigma_v, \quad \sigma_h, \quad C_n, \quad S_n, \quad i$$

Recall that σ_v, σ_h, C_n, S_n are not just one element each but are representing a collection of elements e.g. for the structure shown in Fig. 2.16. One has following elements:

$$E, \; C_3, \; C_3^2, \; \sigma_v^a, \; \sigma_v^b, \; \sigma_v^c, \; \sigma_h \; C_2^a, \; C_2^b, \; C_2^c, \; S_3, \; S_3^2$$

Note: The multiple usage of same symbols (e.g. C_n, σ_m etc.) should not produce confusions regarding their meanings in a given situation.

By now the language of symmetry has been fully developed and one is in a position to say that a particular structure has particular symmetry by enumerating its symmetry elements.

2.1.2 Examples

Example 1

Consider the spring mass system of Fig. 2.9 or the truss of Fig. 2.10. Each of them is seen to have two symmetry elements, namely.

E and σ.

Under the operation of E the systems do not change at all. Under σ however the left part goes to right and vice-versa.

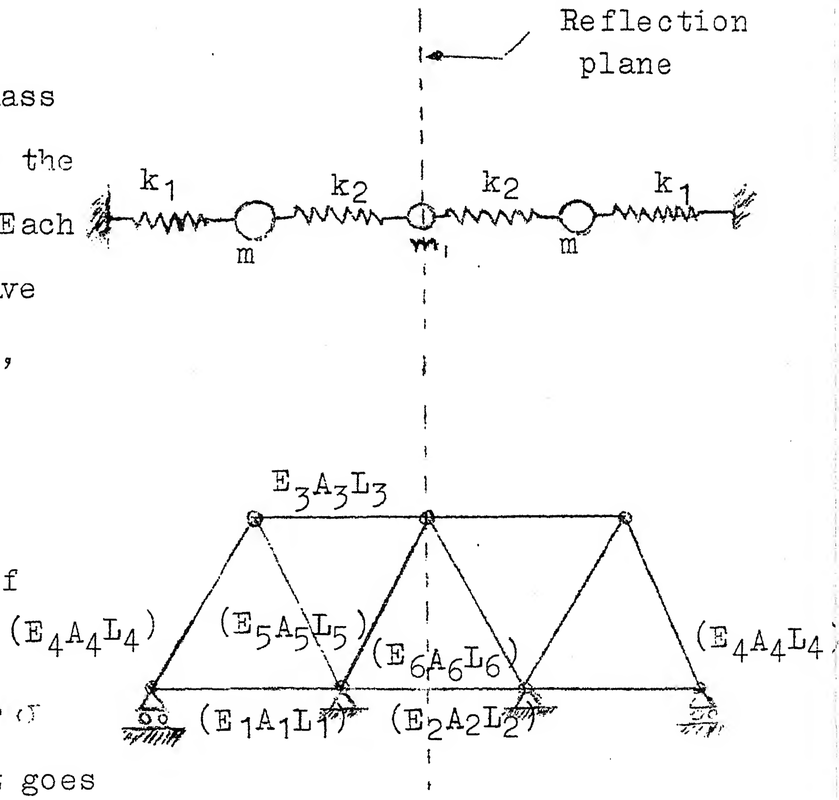


Fig. 2.10

Under σ^2 one gets the original configuration.

Example 2

Consider the system shown in Fig. 2.11, which can be either taken as a cable network or a planner orthogonal grid work. The symmetry elements are

$$E, \ \sigma_1, \ \sigma_2, \ C_2.$$

If the four quadrants are named A,B,C and D then the system can be considered as mnemonic $\begin{array}{|c|c|}\hline A & B \\ \hline C & D \\ \hline\end{array}$ and various symmetry operations are:

$$E \begin{array}{|c|c|}\hline A & B \\ \hline C & D \\ \hline\end{array} = \begin{array}{|c|c|}\hline A & B \\ \hline C & D \\ \hline\end{array}, \qquad \sigma_2 \begin{array}{|c|c|}\hline A & B \\ \hline C & D \\ \hline\end{array} = \begin{array}{|c|c|}\hline B & A \\ \hline D & C \\ \hline\end{array}$$

$$\sigma_1 \begin{array}{|c|c|}\hline A & B \\ \hline C & D \\ \hline\end{array} = \begin{array}{|c|c|}\hline C & D \\ \hline A & B \\ \hline\end{array}, \qquad C_2 \begin{array}{|c|c|}\hline A & B \\ \hline C & D \\ \hline\end{array} = \begin{array}{|c|c|}\hline D & C \\ \hline B & A \\ \hline\end{array}$$

Example 3

Consider now the spring mass system and frame shown in Fig. (2.12) and Fig. (2.13) respectively. Both of the systems have same symmetry elements; namely:

$$E, \ C_3, \ C_3^2, \ \sigma_1, \ \sigma_2 \text{ and } \sigma_3.$$

Let us represent the structure by mneonic (A,B,C) where A, B, and C are the name given to the three

nodes. Then various symmetry operations can be represented as:

$$E\,(A\ B\ C) = (A\ B\ C),\quad C_3(A\ B\ C) = (C\ A\ B)$$

$$C_3^2(A\ B\ C) = (B\ C\ A),\quad \sigma_1(A\ B\ C) = (A\ C\ B)$$

$$\sigma_2(A\ B\ C) = (C\ B\ A),\quad \sigma_3(A\ B\ C) = (B\ A\ C)$$

Example 4:

Consider the 13-masses 42 springs system and the hexagonal space frame shown in Figs. (2.14) and (2.15) respectively. As one can see that these two systems have similar symmetry elements namely:

$$E,\ C_6^1,\ C_6^2,\ C_6^3,\ C_6^4,\ C_6^5,\ \sigma_1,\ \sigma_2,\ \sigma_3,\ \sigma_4,\ \sigma_5,\ \text{and}\ \sigma_6.$$

where E is identity transformation, C_6^m (m=1,2,3,4,5) are the rotations by $m\,\frac{2\pi}{6}$ and σ_k are the reflections across the reflection planes σ_k (k = 1,2, - - - 6).

If the structure is represented by the mnemonic (A_1 A_2 A_3 A_4 A_5 A_6).

where A_i are the corner nodes (i=1, 2,3, - - - 6), then the various symmetry operations can be represented

by following:

	$(A_1\ A_2\ A_3\ A_4\ A_5\ A_6)$
E	$(A_1\ A_2\ A_3\ A_4\ A_5\ A_6)$
C_6	$(A_6\ A_1\ A_2\ A_3\ A_4\ A_5)$
C_6^2	$(A_5\ A_6\ A_1\ A_2\ A_3\ A_4)$
C_6^3	$(A_4\ A_5\ A_6\ A_1\ A_2\ A_3)$
C_5^4	$(A_3\ A_4\ A_5\ A_6\ A_1\ A_2)$
C_6^5	$(A_2\ A_3\ A_4\ A_5\ A_6\ A_1)$

	$(A_1\ A_2\ A_3\ A_4\ A_5\ A_6)$
σ_1	$(A_1\ A_6\ A_5\ A_4\ A_3\ A_2)$
σ_2	$(A_2\ A_1\ A_6\ A_5\ A_4\ A_3)$
σ_3	$(A_3\ A_2\ A_1\ A_6\ A_5\ A_4)$
σ_4	$(A_4\ A_3\ A_2\ A_1\ A_6\ A_5)$
σ_5	$(A_5\ A_4\ A_3\ A_2\ A_1\ A_6)$
σ_6	$(A_6\ A_5\ A_4\ A_3\ A_2\ A_1)$

Example 5

Finally the frame of Fig. (2.16) is considered. This has following symmetry elements as can be seen from the figure itself:

$$E,\ C_3,\ C_3^2,\quad \sigma_v^a,\ \sigma_v^b,\quad \sigma_v^c,\quad \sigma_h\ C_2^a,\ C_2^c,\ S_3,\ S_3^2.$$

where the first six elements have same meaning as that of example 3. σ_h is the reflection across a plane perpendicular to the axis C_3 (the principal axis); C_2^a, C_2^b and C_2^c are

two fold rotations about the axes C_2^a, C_2^b, C_2^c which pass through the middle point of the plane before the members AA', BB', CC' respectively. S_3 and S_3^2 are improper rotations given by $\sigma_h C_3$ and $\sigma_h \cdot C_3^2$ respectively.

Various symmetry operations can be seen to be the following:

	$\begin{pmatrix}A & B & C\\ A' & B' & C'\end{pmatrix}$		$\begin{pmatrix}A & B & C\\ A' & B' & C'\end{pmatrix}$		$\begin{pmatrix}A & B & C\\ A' & B' & C''\end{pmatrix}$		$\begin{pmatrix}A & B & C\\ A' & B' & C'\end{pmatrix}$
E	$\begin{pmatrix}A & B & C\\ A' & B' & C'\end{pmatrix}$	σ_v^a	$\begin{pmatrix}A & C & B\\ A' & C' & B'\end{pmatrix}$	C_2^a	$\begin{pmatrix}A' & C' & B'\\ A & C & B\end{pmatrix}$	σ_h	$\begin{pmatrix}A' & B' & C'\\ A & B & C\end{pmatrix}$
C_3	$\begin{pmatrix}C & A & B\\ C' & A' & B'\end{pmatrix}$	σ_v^b	$\begin{pmatrix}C & B & A\\ C' & B' & A'\end{pmatrix}$	C_2^b	$\begin{pmatrix}C' & B' & A'\\ C & B & A\end{pmatrix}$	S_3	$\begin{pmatrix}C' & A' & B'\\ C & A & B\end{pmatrix}$
C_3^2	$\begin{pmatrix}B & C & A\\ B' & C' & A'\end{pmatrix}$	σ_v^c	$\begin{pmatrix}B & A & C\\ B' & A' & C'\end{pmatrix}$	C_2^c	$\begin{pmatrix}B' & A' & C'\\ B & A & C\end{pmatrix}$	S_3^2	$\begin{pmatrix}B' & C' & A'\\ B & C & A\end{pmatrix}$

where $\begin{pmatrix}A & B & C\\ A' & B' & C'\end{pmatrix}$ is the mnemonic of the structure.

If one chooses any two elements from symmetry elements and multiplies them (in the sense that in operation one follows the other) the result is one of the element from the set e.g.

$$\sigma_h \cdot \sigma_v^a = C_2^a \qquad C_2^a \cdot C_3 = C_2^b \qquad S_3 \cdot \sigma_h = C_3 .$$

$$\sigma_h \cdot \sigma_v^b = C_2^b \quad \text{and} \quad C_3 \cdot C_2^a = C_2^c \qquad S_3^2 \cdot \sigma_v^c = C_2^a$$

etc. etc. etc.

Thus symmetry elements are only and the only things which describe the symmetry of any system because any number of products of these elements is again one of these. It can be seen that under the operation of all the above symmetry elements (except for the translation) at least one point remains fixed and hence the symmetry constituting them is called point symmetry.

So far it has been implicitly assumed that if there is geometrical or configurational symmetry then corresponding symmetries hold true for mechanical behaviour of members also, (e.g. see Figs. 2.9 - 2.16). It is not clear as to under what circumstances and in what sense one can have structural symmetry even without geometrical symmetry e.g. see Fig. 2.17. The two beams may be structurally symmetric (in a restricted sense to be defined later) even though they are neither geometrically nor materially symmetric. To clearify the matter, one needs a procedure to seperate out the geometry

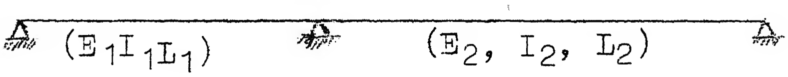


Fig. 2.17

and topology from members properties and for that purpose the next two articles are needed.

2.2 SYMMETRIES OF MATRICES:

Consider a n x n matrix K.

The symmetry, $K = K^T$ is known to every one. If this matrix is stiffness matrix for a structural system and the structural system has certain symmetry elements then those symmetry element will show up in K also, provided proper numbering of nodal degrees of freedom and proper co-ordinate systems are chosen. This is where the topological and geometrical property of structure seems to play a role. As an example consider the cantilever shown in Fig. 2.18. There are 2n-degrees of freedom and hence the flexibility matrix "A" is of the order 2n x 2n.

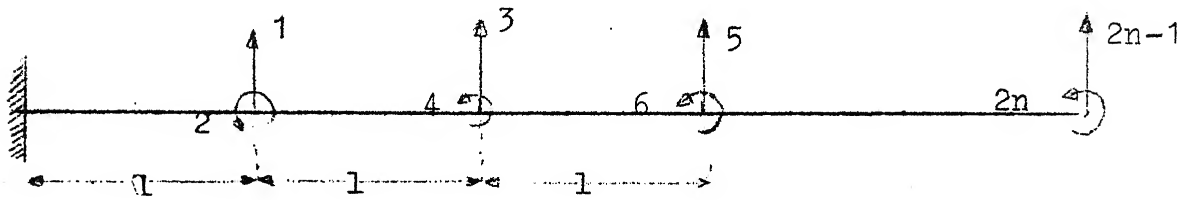


Fig. 2.18

i.e

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & - - - - & A_{1n} \\ A_{21} & A_{22} & A_{23} & - - - - & A_{2n} \\ A_{31} & A_{32} & A_{33} & - - - - - & A_{3n} \\ - & - & - & - - - - & - \\ A_{n1} & A_{n2} & A_{n3} & - - - - & A_{nn} \end{bmatrix}$$

where A_{ij} are 2 x 2 matrices and i, j represent the node number and $A_{ij} = A^T_{ji}$ from Green-Maxwell reciprocal theorem. It can be shown that only one matrix out of $\frac{n(n+1)}{2}$ matrices need to be known and all others are generated from that one, viz:

$$A_{jj} = A_{11}(jl)$$

$$j = 1, 2, ---, n$$

$$A_{11}(l) = \frac{1}{EI} \begin{bmatrix} l^2/2 & l/2 \\ l/2 & 1 \end{bmatrix}$$

$$A_{ij} = T^{i-j} A_{jj} \quad (j < i) \quad i, j = 1, 2, 3, ---, n$$

Where $T = \begin{bmatrix} 1 & l \\ 0 & 1 \end{bmatrix}$

Also it can be seen that,

$$T^m = \begin{bmatrix} 1 & ml \\ 0 & 1 \end{bmatrix}$$

This sort of symmetry in flexibility matrix is due to the repetitions of similar members. Algorithm can be developed to ease the numerical computations for such matrices. Similarly other symmetries in stiffness or flexibility matrix can occur if the structural system has some other symmetry. The problem however, for the time being is reverse one; i.e. to identify the structural symmetry, when the stiffness matrix is given to have some symmetry. The scheme is not unique but still useful because in analysis and sometimes in design only this part is known to play almost every role. To design a mechanical system with given natural frequencies is an example to such types of problems. This type of problem is encountered in design under random excitations and also for shock absorption. With these motives the various types of matrix symmetries will be discussed below. There are two types of symmetries. Gross symmetries and internal symmetries.

Let $K = n \times n$ matrix.

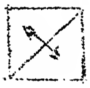
Following examples are given for gross symmetries.

Example 1

K is said to be counter diagonal symmetric when

$$K = C\,K^T C \;=\; K^c$$

where C = counter identity =
$$\begin{bmatrix} & & & & 1 \\ & O & & 1 & \\ 0 & & \cdot & & \\ & \cdot & & O & \\ 1 & & & & \end{bmatrix}$$

This symmetry can be represented by: 

Example 2

K is said to be centrosymmetric if

$$K = CKC$$

Thus $K = K^c$ and $K = K^T \Rightarrow$ centro symmetry,

represented as: 

Theorem:

If K is centro-symmetric then by proper inter-charges of rows and column it can be written as

$$K = \begin{bmatrix} A & B \\ B & A \end{bmatrix}$$

provided $K = K^T$
where A and B are $n/2 \times n/2$ matrices
and $A = A^T$, $B = B^T$

Note that A and B may have symmetries similar to that of K. Proof of the theorem is simple and is illustrated as below. Let K = 4 x 4 matrix, $K = K^T$ and it is centrosymmetric, i.e.

$$K = \begin{bmatrix} a & b & c & d \\ b & e & f & c \\ c & f & e & b \\ d & c & b & a \end{bmatrix} \text{ can be written as: } \left[\begin{array}{cc|cc} a & b & c & d \\ b & e & f & c \\ \hline c & d & a & b \\ f & c & b & e \end{array}\right] = \begin{bmatrix} A & B \\ B & A \end{bmatrix}$$

Example 3

K is skew counterdiagonal symmetric if

$$K = -\ CK^TC = -\ K^C$$

Example 4

K is skew centrosymmetric if

$$K = -\ CKC$$

Theorem

If K is skew centro-symmetric it can be written as

$$K = \begin{bmatrix} A & B \\ -B & -A \end{bmatrix}$$ provided $K = K^T$ and then $A = A^T$, $B = B^T$

Example 5

K is cyclically symmetric if it can be written as

$$K = \begin{bmatrix} A_1 & A_2 & --- & A_m \\ A_m & A_1 & --- & A_{m-1} \\ - & --- & & --- \\ A_2 & A_3 & --- & A_1 \end{bmatrix} \quad \text{OR} \quad K = \begin{bmatrix} A_1 & A_2 & A_3 & --- & A_m \\ A_2 & A_3 & A_4 & --- & A_1 \\ A_3 & A_4 & A_5 & --- & A_2 \\ - & - & - & - & - \\ A_m & A_1 & A_2 & --- & A_{m-1} \end{bmatrix}$$

where $A_1, A_2, \;---\; A_m$ are sub-matrices of order $\frac{n}{m}$.

Note: The both definitions are one and the same and such matrices will be seen to be the stiffness matrices of cyclically symmetric structures in Chapter III.

Example 6

K is centro-cyclically symmetric if it can be written as:

$$K = \left[\begin{array}{cccc|cccc} A_1 & A_2 & ---- & A_m & B_1 & B_2 & ---- & B_m \\ A_m & A_1 & ---- & A_{m-1} & B_m & B_1 & ---- & B_{m-1} \\ - & - & - & - & - & - & - & - \\ A_2 & A_3 & ---- & A_1 & B_2 & B_3 & ---- & B_1 \\ \hline B_1 & B_2 & ---- & B_m & A_1 & A_2 & ---- & A_m \\ B_m & B_1 & --- & B_{m-1} & A_m & A_1 & --- & A_{m-1} \\ - & - & - & - & - & - & - & - \\ B_2 & B_3 & ---- & B_1 & A_2 & A_3 & ---- & A_1 \end{array}\right]$$

where A_i and B_i are $\frac{n}{2m} \times \frac{n}{2m}$ matrices.

Example 7

K is cyclic-centro-symmetric if it can be written as

$$K = \begin{bmatrix} \begin{bmatrix} A_1 & B_1 \\ B_1 & A_1 \end{bmatrix} & \begin{bmatrix} A_2 & B_2 \\ B_2 & A_2 \end{bmatrix} & \cdots & \begin{bmatrix} A_m & B_m \\ B_m & A_m \end{bmatrix} \\ \begin{bmatrix} A_m & B_m \\ B_m & A_m \end{bmatrix} & \begin{bmatrix} A_1 & B_1 \\ B_1 & A_1 \end{bmatrix} & \cdots & \begin{bmatrix} A_{m-1} & B_{m-1} \\ B_{m-1} & A_{m-1} \end{bmatrix} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \begin{bmatrix} A_2 & B_2 \\ B_2 & A_2 \end{bmatrix} & \begin{bmatrix} A_3 & B_3 \\ B_3 & A_3 \end{bmatrix} & \cdots & \begin{bmatrix} A_1 & B_1 \\ B_1 & A_1 \end{bmatrix} \end{bmatrix}$$

Note: Symmetries of types 6 and 7 will be encountered in Chapter III inconnection with cyclic-Reflection symmetric structures.

Theorem: If a matrix is cyclically centro-symmetric then it can be centro-cyclic symmetric and vice-versa.

Proof: Consider the above matrix and let us place the 2nd row at the m + 1st row, 4th row at m + 2nd row etc. and at the same time third

row at the 2nd and 5th row at the 3rd and so on then the following form is obtained.

$$K' = \begin{bmatrix} A_1 & B_1 & A_2 & B_2 & ---- & A_m & B_m \\ A_m & B_m & A_1 & B_1 & ---- & A_{m-1} & B_{m-1} \\ A_{m-1} & B_{m-1} & A_m & B_m & ---- & A_{m-2} & B_{m-2} \\ - & - & - & - & - & - & - \\ A_2 & B_2 & A_3 & B_3 & --- & A_1 & B_1 \\ B_m & A_m & B_1 & A_1 & ---- & B_{m-1} & A_{m-1} \\ - & - & - & - & - & - & - \\ B_2 & A_2 & B_3 & A_3 & -- & -B_1 & A_1 \end{bmatrix}$$

If similar operations are performed over columns one gets:

$$K = \left[\begin{array}{ccccc|ccccc} A_1 & A_2 & A_3 & --- & A_m & B_1 & B_2 & --- & & B_m \\ A_m & A_1 & A_2 & --- & A_{m-1} & B_m & B_1 & --- & & B_{m-1} \\ - & - & - & - & - & - & - & - & & - \\ A_2 & A_3 & ---- & & A_1 & B_2 & B_3 & ---- & & B_1 \\ \hline B_1 & B_2 & ---- & & B_m & A_1 & A_2 & --- & & A_m \\ B_m & B_1 & ---- & & B_{m-1} & A_m & A_1 & --- & & A_{m-1} \\ - & - & - & - & - & - & - & - & & - \\ B_2 & B_3 & ---- & & B_1 & A_2 & A_3 & --- & & A_1 \end{array}\right] = \text{Centro-cyclic symmetric}$$

The proof of the second part is obtained on exactly similar lines.

Note: If $K = K^T$ then in all the above symmetries there is additional symmetry which reduces the number of independent elements still further.

Furthermore all the symmetries correspond to invariance of K under some set of similarity transformations.

Internal Symmetries:

Internal symmetries are the one in which the elements or submatrices of a matrix are given by some functions of a set of elements or submatrices of the matrix. This definition is fairly general and hence it is not possible to enumerate these symmetries as has been done for gross symmetries. Examples of the internal symmetries are following:

Example 1

$$K = \begin{bmatrix} K_{11} & K_{12} & K_{13} & - & - & - & - & K_{1m} \\ K_{21} & K_{22} & K_{23} & - & - & - & - & K_{2m} \\ - & - & - & - & - & - & & - \\ K_{m1} & K_{m2} & K_{m3} & - & - & - & & K_{mm} \end{bmatrix}$$

where

$$K_{11}(l) = \frac{1}{EI}\begin{bmatrix} l^2/3 & l/2 \\ l/2 & 1 \end{bmatrix}$$

$$K_{jj} = K_{11}\,(jl)$$

$$K_{ij} = T^{i-j}\,A_{jj} = K^T_{ji} \qquad (j<i)$$

$$T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

This matrix has been encountered earlier.

<u>Example 2</u>

$$\begin{bmatrix} K_1 & K_2 & K_3 & ---- & K_m \\ K_1\theta_1 & K_2\theta_2 & K_3\theta_3 & --- & K_m\theta_m \\ K_1\theta_1^2 & K_2\theta_2^2 & K_3\theta_3^2 & --- & K_m\theta_m^2 \\ - & - & - & - & - \\ K_1\theta_1^{m-1} & K_2\theta_2^{m-1} & K_3\theta_3^{m-1} & --- & K_m\theta_m^{m-1} \end{bmatrix}$$

Thus the whole matrix is seen to be generated by m- submatrices (K_1, K_2 - - - K_m) of m x m and m - parameters

θ_1 - - - - θ_m. This type of matrix will be encountered in connection with cyclic structure.in Chapter III.

Such symmetries result due to presence of repetitive symmetry in structural systems. In the existing literatures, no alogrithm is found for simplifying the problems by using these symmetries except in some simple cases. Noteworthy is the thing that the gross symmetries discussed earlier are particular cases of this general symmetry the physical reason is that repetitive symmetry is a general case of gross symmetry. As remarked earlier the repetitive symmetry is not given much attention in this work. It will be seen in the next chapter how the stiffness matrices of the structural systems are:

Centro-symmetric, counter diagonal symmetric, skew-centro-symmetric, cyclically symmetric and centro-cyclically or cyclic-centro symmetric, dependendi*n*g upon the symmetry elements of the structurals system. Also it will be seen how these form of the matrices help in getting their inverse, their eigen values and eigen vectors etc.

2.3 CONTRIBUTIONS OF GEOMETRY-TOPOLOGY AND MEMBERS SYMMETRIES TO THE SYMMETRIES OF STIFFNESS MATRIX 'K'

Let R_F be the vector space corresponding to the member end forces and displacements. Let R_P be the vector space corresponding to the joint forces and displacements, where joint means only those joints which have some degree of freedom.

Then the configuration obtained by joining these members forms a structural system only if R_P is a projection of R_F and therefore any vector in R_P is a projection of some vector in R_F. i.e. if $P \in R_P$ then there exists a $F \in R_F$ such that

$$P = A F$$

where A = a rectangular matrix (the projection matrix).

Now let P= joint force vector

X= joint displacement vector

F= members ends force vector

e= members ends displacement vector,

then one will have,

$$P = AF \quad \text{and} \quad X = Be$$

Consider now work done $W = X^TP = e^TF$

Then $e^TB^TAF = e^TF \Rightarrow B^TA = \text{identity matrix} = A^TB$

Let $F = Se$

Then $AF = AS^Te = ASA^TBe$

$$P = ASA^TX$$

$$= K.X$$

Therefore, stiffness matrix $K = ASA^T$

where S = stiffness matrix of the aggregates of the members.

Any symmetry of the aggregate of members will cast itself into S; e.g. if one has "m" members in planner bending then,

$$S = \begin{bmatrix} K_{11} & & & 0 \\ & K_{22} & & \\ & & \ddots & \\ 0 & & & K_{mm} \end{bmatrix}$$

where

$$K_{ii} = \begin{bmatrix} \dfrac{4\,E_iI_i}{L_i} & \dfrac{2E_iI_i}{L_i} \\ \dfrac{2E_iI_i}{L_i} & \dfrac{4E_iI_i}{L_i} \end{bmatrix} \qquad i = 1,2,\ldots,m$$

For members in pin-jointed trusses one has,

$$S = \begin{bmatrix} k_1 & & & 0 \\ & k_2 & & \\ & & \ddots & \\ 0 & & & k_m \end{bmatrix}$$

where $k_i = \dfrac{E_i\ A_i}{L_i}$, will exhibit the symmetry of member's aggregate.

It is now clear that the matrix 'S' has nothing to do with the positions of the individual members in the system while the matrix "A" is solely determined by the geometrical configurations of various members and of course the nodal numbering and co-ordinate systems. But, the symmetries of stiffness matrix K are governed by the symmetries of two matrices A and S. It has been seen in the previous section and will be seen in the

next chapter also, that the gross symmetries of a matrix are always determined by the invariance of the matrix under some sort of similarity transformations, viz.

$$T_i \; K \; T_i^T \quad = \quad K$$

where T_i are some matrices of n x n (orthogonal)

$$(i = 1, 2, \text{----} \; r)$$

Now $K = ASA^T$ (A is n x m matrix).

Therefore, $T_i \; ASA^T T_i^T = K$

Now there are four ways in which the above can be satisfied:

(i) $T_i A = A$ In this case the symmetries of

$A^T T_i^T = A^T$ K are determined purely by geometry and topology.

(ii) There are another set of matrices $\bar{T}_i$ of m x m such that: $(i = 1, 2, \text{---} \; r)$

$$T_i A \bar{T}_i^T = A$$

$$\bar{T}_i A^T T_i^T = A^T$$

and $$\bar{T}_i S \bar{T}_i^T = S$$

Here the symmetries of both A and S play equal role in symmetries of K and hence in structural symmetry.

(iii) Symmetries of S and A are of different order; i.e. the number of T_i and $\bar{T}_i$ are different. In such cases the symmetries of K will depend upon the variables involved in A and S.

(iv) If neither A nor S has any symmetry; (except the well known symmetry of transposition which is not the purpose of discussion) it is possible to get certain symmetries in K depending upon the variables involved. As an example consider the truss shown in Fig. 2.19. Then,

$$\begin{bmatrix} P_1 \\ P_2 \\ \hdashline P_3 \\ P_4 \end{bmatrix} = \left[\begin{array}{cc:cc:c} \cos\alpha_1 & -\cos\alpha_4 & & & -1 \\ \sin\alpha_1 & \sin\alpha_4 & \multicolumn{2}{c}{0} & 0 \\ \hdashline & & \cos\alpha_3 & -\cos\alpha_2 & -1 \\ \multicolumn{2}{c:}{0} & \sin\alpha_3 & \sin\alpha_2 & 0 \end{array}\right] \begin{bmatrix} F_1 \\ F_2 \\ \hdashline F_3 \\ F_4 \\ \hdashline F_5 \end{bmatrix}$$

$$= \left[\begin{array}{c:c:c} A_1 & 0 & a \\ \hdashline 0 & A_2 & a \end{array}\right] \begin{bmatrix} F_1' \\ F_2' \\ F_5 \end{bmatrix}$$

$$\begin{bmatrix} F_1 \\ F_2 \\ \hdashline F_3 \\ F_4 \\ F_5 \end{bmatrix} = \left[\begin{array}{ccccc} \frac{A_1E_1}{L_1} & & 0 & & 0 \\ & \frac{A_2E_2}{L_2} & & & 0 \\ \hdashline & & \frac{A_3E_3}{L_3} & & 0 \\ 0 & & & \frac{A_4E_4}{L_4} & 0 \\ 0 & 0 & 0 & 0 & \frac{A_5E_5}{L_5} \end{array}\right] \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{bmatrix}$$

$$= \left[\begin{array}{cc:c} B_1 & 0 & 0 \\ \hdashline 0 & B_2 & 0 \\ \hdashline 0 & 0 & b \end{array}\right] \begin{bmatrix} e^1 \\ e^2 \\ e^5 \end{bmatrix}$$

$$\text{Therefore} \quad \bar{A}S\bar{A}^T = \begin{bmatrix} A_1 & 0 & a \\ 0 & A_2 & a \end{bmatrix} \begin{bmatrix} B_1 & 0 & 0 \\ 0 & B_2 & 0 \\ 0 & 0 & b \end{bmatrix} \begin{bmatrix} A_1^T & 0 \\ 0 & A_2^T \\ a^T & a^T \end{bmatrix}$$

$$= \begin{bmatrix} A_1B_1A_1^T + baa^T & baa^T \\ baa^T & A_2B_2A_2^T + baa^T \end{bmatrix}$$

$$\text{or} \quad K = \bar{A}S\bar{A}^T = \begin{bmatrix} K_1 & K_2 \\ K_2 & K_1 \end{bmatrix} \quad \text{if}$$

$$A_1B_1A_1^T + baa^T = A_2B_2A_2^T + baa^T$$

$$\text{i.e.} \quad A_1B_1A_1^T = A_2B_2A_2^T \quad (\det|A_2| = \sin(\alpha_2+\alpha_3) \neq 0)$$

$$B_2 = A_2^{-1} \; A_1B_1 \; A_1^T \; A_2^{T-1}$$

$$= A_2^{-1} \; A_1 \; B_1 \; (A_2^{-1} \; A_1)^T$$

$$A_2^{-1} = \frac{1}{\text{Sin}(\alpha_2+\alpha_3)} \begin{bmatrix} \text{Sin}\,\alpha_2 & \text{Cos}\,\alpha_2 \\ -\text{Sin}\,\alpha_3 & \text{Cos}\,\alpha_3 \end{bmatrix}$$

So $$A_2^{-1} A_1 = \frac{1}{\operatorname{Sin}(\alpha_2+\alpha_3)} \begin{bmatrix} \operatorname{Sin}\alpha_2 & \operatorname{Cos}\alpha_2 \\ -\operatorname{Sin}\alpha_3 & \operatorname{Cos}\alpha_3 \end{bmatrix} \begin{bmatrix} \operatorname{Cos}\alpha_1 & -\operatorname{Cos}\alpha_4 \\ \operatorname{Sin}\alpha_1 & \operatorname{Sin}\alpha_4 \end{bmatrix}$$

$$= \frac{1}{\operatorname{Sin}(\alpha_2+\alpha_3)} \begin{bmatrix} \operatorname{Sin}(\alpha_1+\alpha_2) & \operatorname{Sin}(\alpha_4-\alpha_2) \\ \operatorname{Sin}(\alpha_1-\alpha_3) & \operatorname{Sin}(\alpha_3+\alpha_4) \end{bmatrix}$$

$$(A_2^{-1} A_1)^T = \begin{bmatrix} \operatorname{Sin}(\alpha_1+\alpha_2) & \operatorname{Sin}(\alpha_1-\alpha_3) \\ \operatorname{Sin}(\alpha_4-\alpha_2) & \operatorname{Sin}(\alpha_3+\alpha_4) \end{bmatrix} \frac{1}{\operatorname{Sin}(\alpha_2+\alpha_3)}$$

$$B_2 = \begin{bmatrix} \frac{E_3A_3}{L_3} & 0 \\ 0 & \frac{E_4A_4}{L_4} \end{bmatrix}$$

$$= \frac{1}{\operatorname{Sin}^2(\alpha_2+\alpha_3)} \begin{bmatrix} \operatorname{Sin}(\alpha_1+\alpha_2) & \operatorname{Sin}(\alpha_4-\alpha_2) \\ \operatorname{Sin}(\alpha_1-\alpha_3) & \operatorname{Sin}(\alpha_3+\alpha_4) \end{bmatrix} \begin{bmatrix} \frac{E_1A_1}{L_1} & 0 \\ 0 & \frac{E_2A_2}{L_2} \end{bmatrix} \times$$

$$\begin{bmatrix} \operatorname{Sin}(\alpha_1+\alpha_2) & \operatorname{Sin}(\alpha_1-\alpha_3) \\ \operatorname{Sin}(\alpha_4-\alpha_2) & \operatorname{Sin}(\alpha_3+\alpha_4) \end{bmatrix}$$

or $$\begin{bmatrix} \frac{E_3A_2}{L_3} \\ \frac{E_4A_4}{L^4} \end{bmatrix} = \frac{1}{s_{23}^2} \begin{bmatrix} s_{12}^2 & \bar{s}_{42}^2 \\ \bar{s}_{13}^2 & s_{34}^2 \end{bmatrix} \begin{bmatrix} \frac{E_1A_1}{L_1} \\ \frac{E_2\ A_2}{L_2} \end{bmatrix}$$

and $$\frac{E_1A_1}{L_1} \quad s_{12}\ \bar{s}_{13} + \frac{E_2\ A_2}{L_2}\ \bar{s}_{34}\ \bar{s}_{42} = 0$$

where $s_{ij} = \text{Sin}\ (\alpha_i + \alpha j)$

$$\bar{s}_{ij} = \text{Sin}\ (\alpha_i - \alpha j)$$

or $$\frac{E_3A_3}{L_3} = \frac{E_1A_1}{L_1}\ \frac{s_{12}^2}{s_{23}^2}\ \left(1 + \frac{\bar{s}_{13}^2}{s_{34}^2}\ \frac{E_1A_1}{E_2A_2}\ \frac{L_2}{L_1}\right)$$

$$\frac{E_4A_4}{L_4} = \frac{E_1A_1}{L_1}\ \frac{\bar{s}_{13}^2}{s_{23}^2}\ \left(1 + \frac{s_{34}^2}{\bar{s}_{13}^2}\ \frac{E_2A_2}{E_1A_1}\ \frac{L_1}{L_2}\right)$$

These equations can be easily satisfied by proper choice of $\frac{E_1A_1}{L_1}$, $\frac{E_2A_2}{L_2}$, and $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ because all the them can be varied independently except one constraint i.e.

$$L_2^2 + L_3^2 - 2L_2\ L_3\ \text{Cos}\ (\alpha_3 - \alpha_4) = L_1^4 + L_4^2 - 2L_1L_4\text{Cos}(\alpha_1 - \alpha_2) = L_5^2$$

Thus

$$K = \begin{bmatrix} A_1B_1A_1^T & baa^T \\ baa^T & A_1B_1A_1^T \end{bmatrix}$$

= a centrosymmetric matrix or a

cyclically symmetric matrix even though there is no trace of any symmetry in geometry and members aggregate. Such symmetries may be called incidental symmetries.

Possibilities (i), (ii) and (iv) are available under what circumstances, is not the purpose of the present work. The purpose was only to show, the contributions of geometry, topology and members aggregate symmetries to structural symmetry. In what follows, the possibility (iii) is assumed and it is only in this spirit the symmetry of first section of this chapter was referred.

In what follows, the structural problem is meant only for elastic structures and usually linear ones.

2.4 LOAD AND STRUCTURAL SYSTEMS

So far load has not yet come into picture. The reason for this is that any linear physical problem is represented by:

$$LX = F$$

where L = a linear operator, (Matrix operator, Differential operator, Integral operator etc.)

X = response or output, (a vector, a function or a set of functions

F = excitation or input (a vector, a function, or a set of functions etc.)

Thus F comes only on the right hand side and has nothing to do with L.

The solution of the problem is at once obtained, if L^{-1} is obtained. Thus the basic problem is to obtain the L^{-1} The inversion is simplified if it can be factorised. e.g. if $L = \nabla^2$ and if the problem has spherical symmetry (Excitation has nothing to do with the symmetry of L) then if r, θ, ϕ co-ordinates are chosen, the seperation is at once applicable and the solutions obtained are liable to satisfy the boundary conditions i.e. ∇^2 gets factorised into r, θ, ϕ and hence the inversion is simplified. A similar argument is applied for any operator. If proper co-ordinates, the so-called symmetry co-ordinates are chosen the operator L gets factorised. If L is a matrix operator, the factorisation of L means its block diagonalisation and which is basically the purpose of the following chapters.

To this end, one is able to state the structural symmetry in following form:

Let $\{S\}$ be the set of symmetry element of the system. Then L will have same symmetries.

Let $\{P\}$ be the set of elements of symmetry of load system acting on the structure.

Then if $\{S\}$ and $\{P\}$ are same then the excitations X will also have same symmetries and one is atonce in a position to judge the number of independent excitation components. If $\{S\} \cap \{P\} = \emptyset$ then there can not be any symmetry in excitation X and then one can use the elements of S only to simplify the inversion of L etc.

So far the boundary conditions have not explicitly come into picture but they have been implicitly assumed to contribute to/the symmetry of the systems along with the geometry etc. of the system.

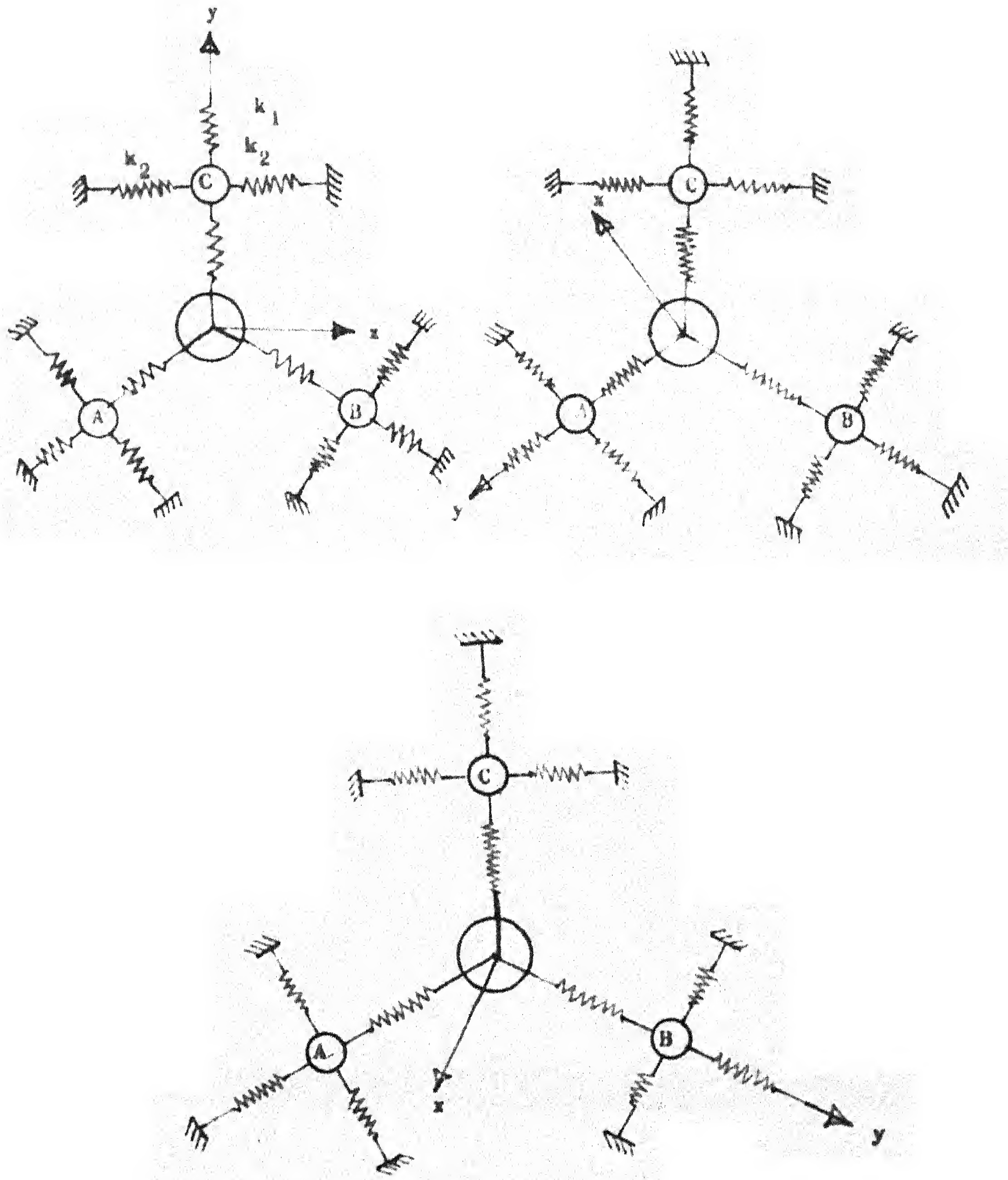


FIG. 2.1 SPRING MASS SYSTEM WITH SYMMETRY ELEMENT
E, C_3, C_3^2, σ_1, σ_2, and σ_3

HERE THE CO-ORDINATE SYSTEM IS ROTATED OR OPERATED BY THE SYMMETRY ELEMENTS. THIS DESCRIPTION OF THE MECANICAL SYSTEM IS EQUIVALENT TO THAT OF FIG 2.2

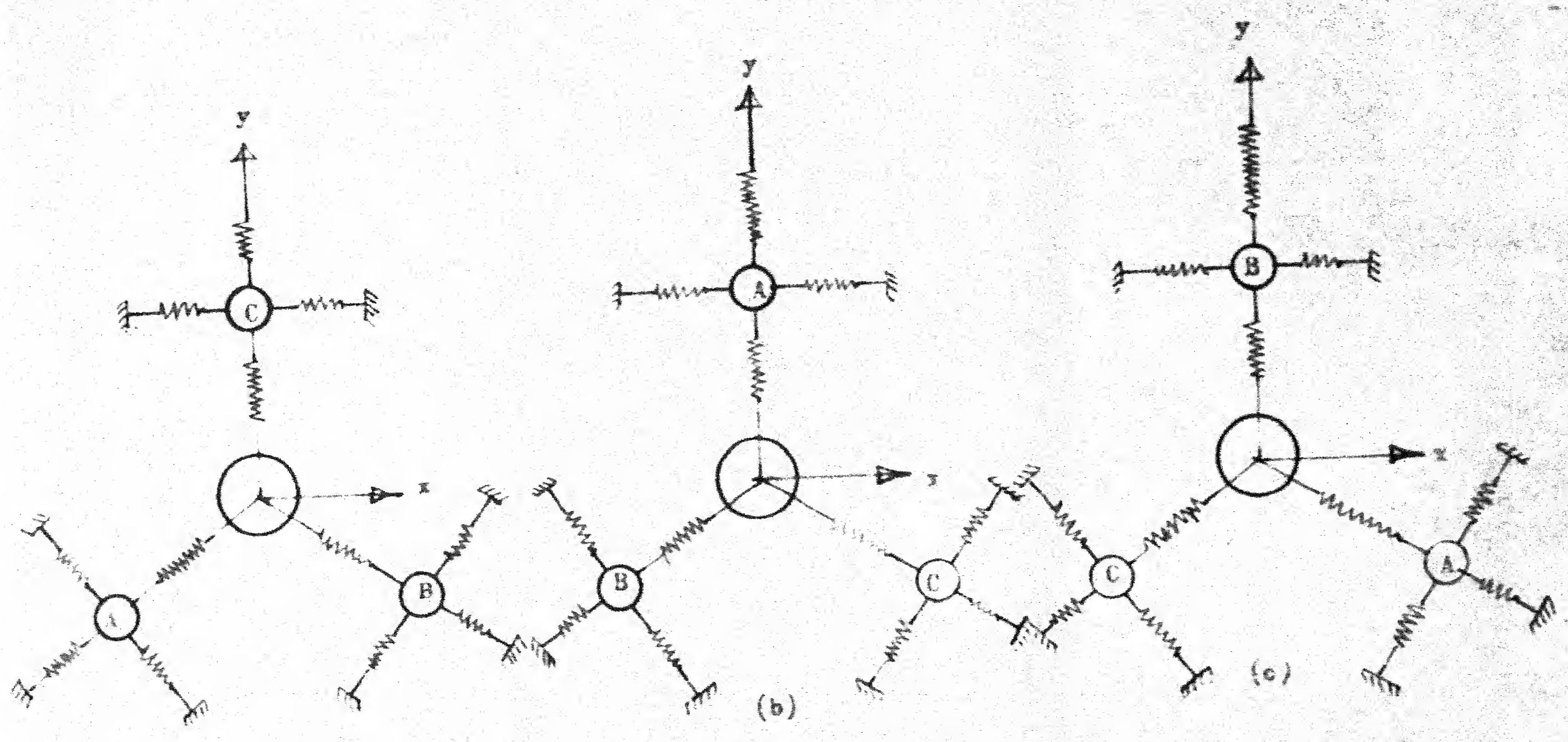


FIG. 2.2

(a)

SPRING MASS SYSTEM WITH SYMMETRY ELEMENTS E, C_3, C_3^2 σ_1, σ_2, and σ_3
AND EXACTLY SIMILAR TO THAT OF FIGURE 2.1

NOTE: σ_1, σ_2 AND σ_3 ARE σ_v^a, σ_v^b AND σ_v^c RESPECTIVELY IF ONE CONSIDERS THE REFLECTIONPLANE σ_h ALSO.

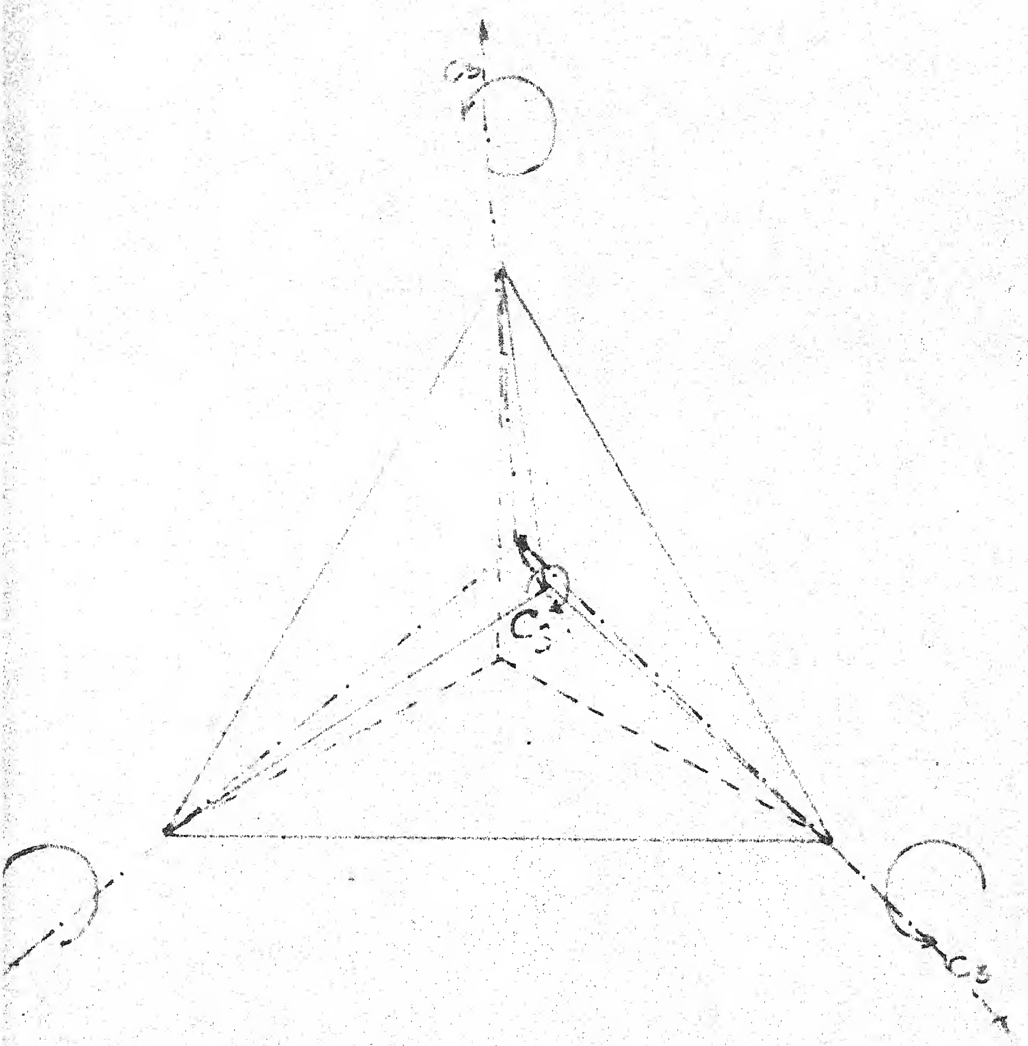


Fig. 2·3

TETRAHEDRAL FRAME WITH FOUR 3-FOLD AXES OF SYMMETRY.

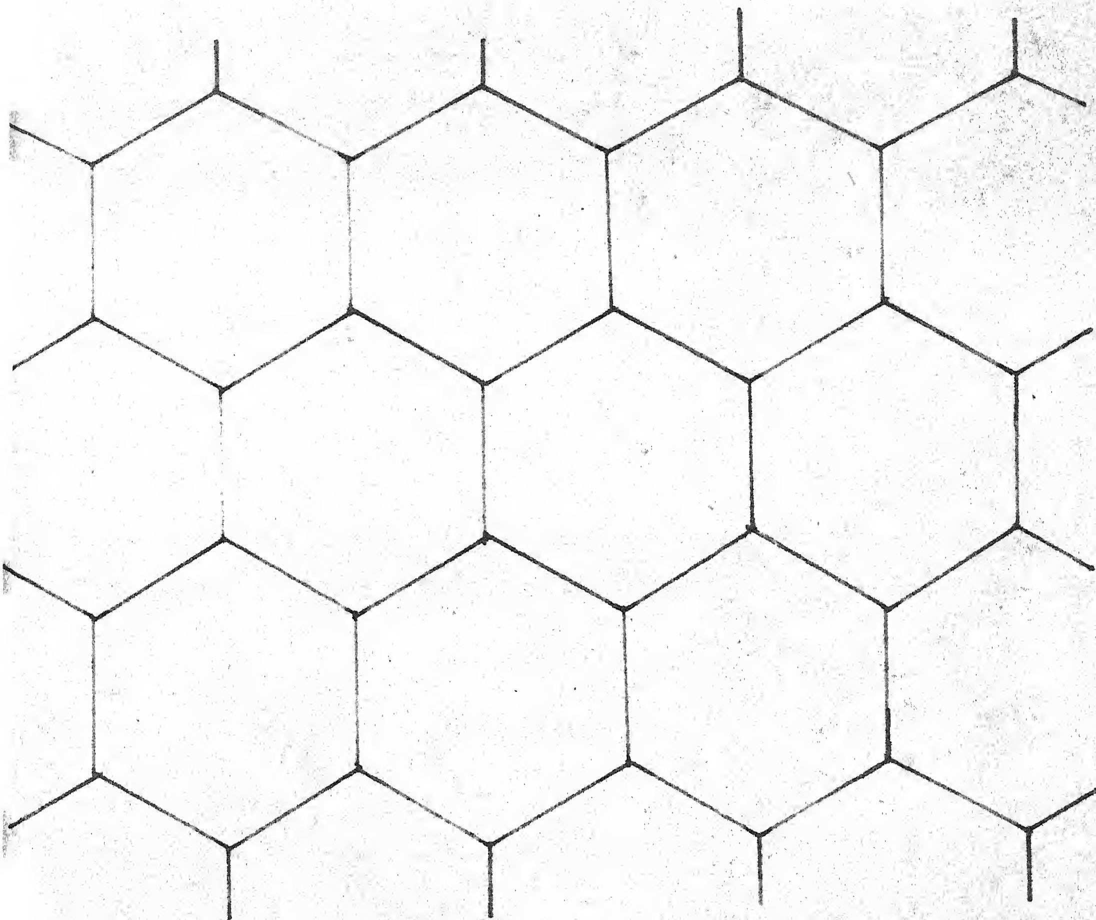


FIG. 2.7

TRANSLATIONALLY SYMMETRIC HEXAGONAL GRID-
-WORK.

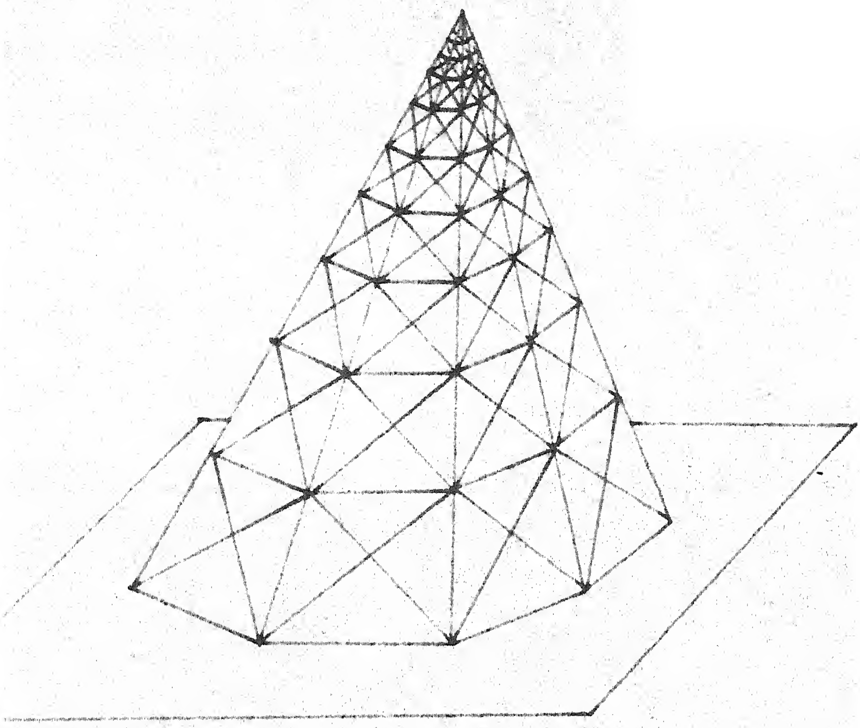


FIG. 2.8

MAGNIFIED TRANSLATIONAL SYMMETRY, ASSUMING I OF INFINITE EXTENT IN ONE DIRECTION.

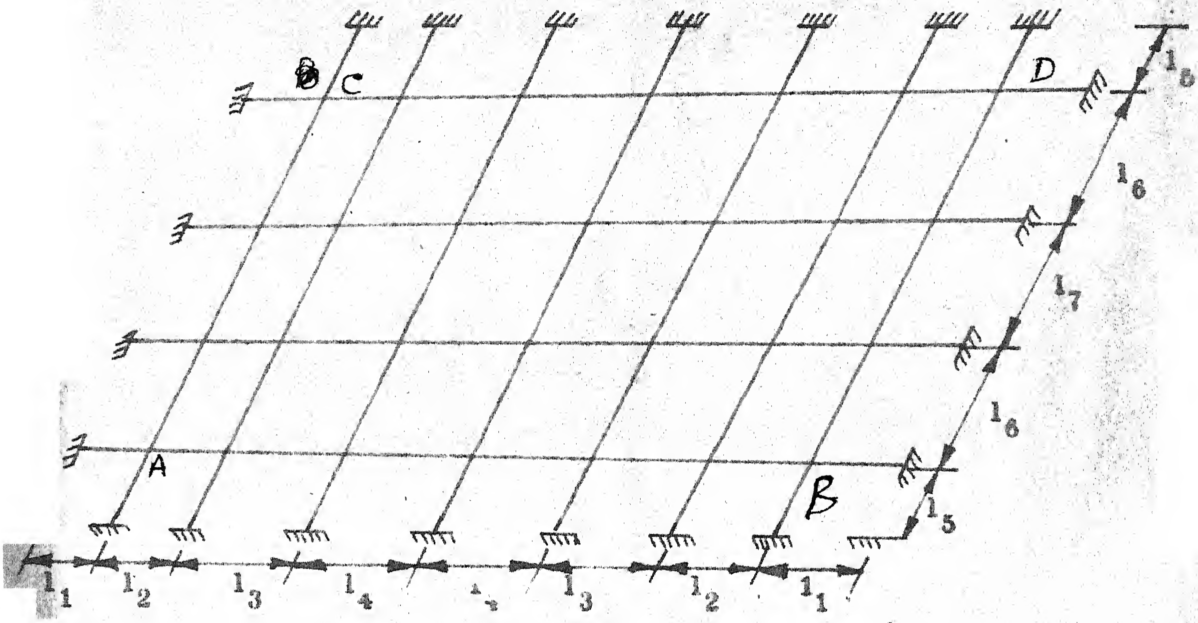
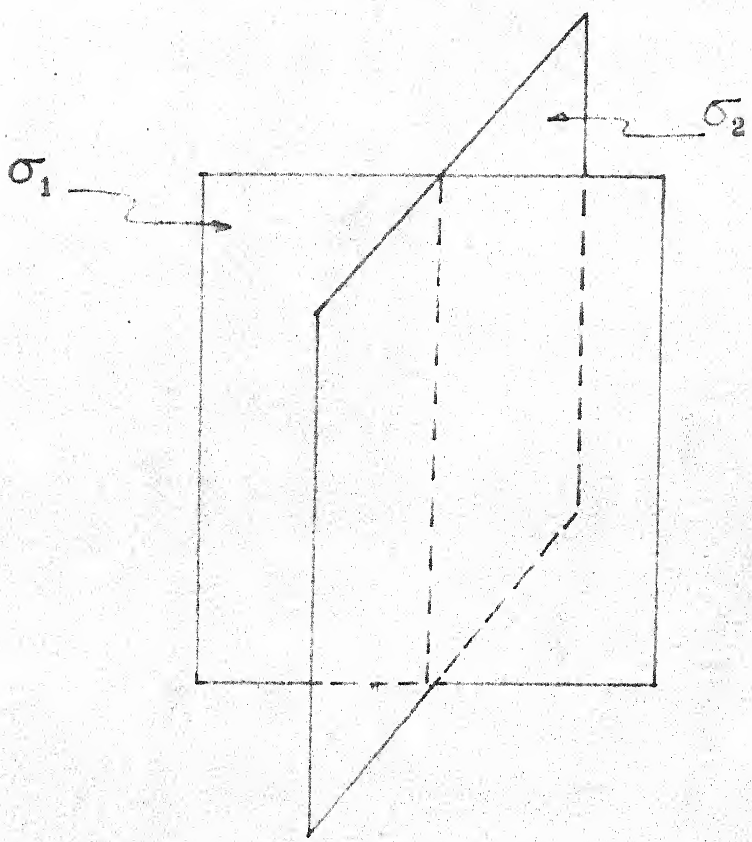


FIG. 2.11

THE ORTHOGONAL GRID-WORK OR CABLE NETWORK WITH SYMMETRY ELEMENTS E, σ_1, σ_2, and C_2.



THE TWO SYMMETRY PLANES OF THE ABOVE SYSTEM.

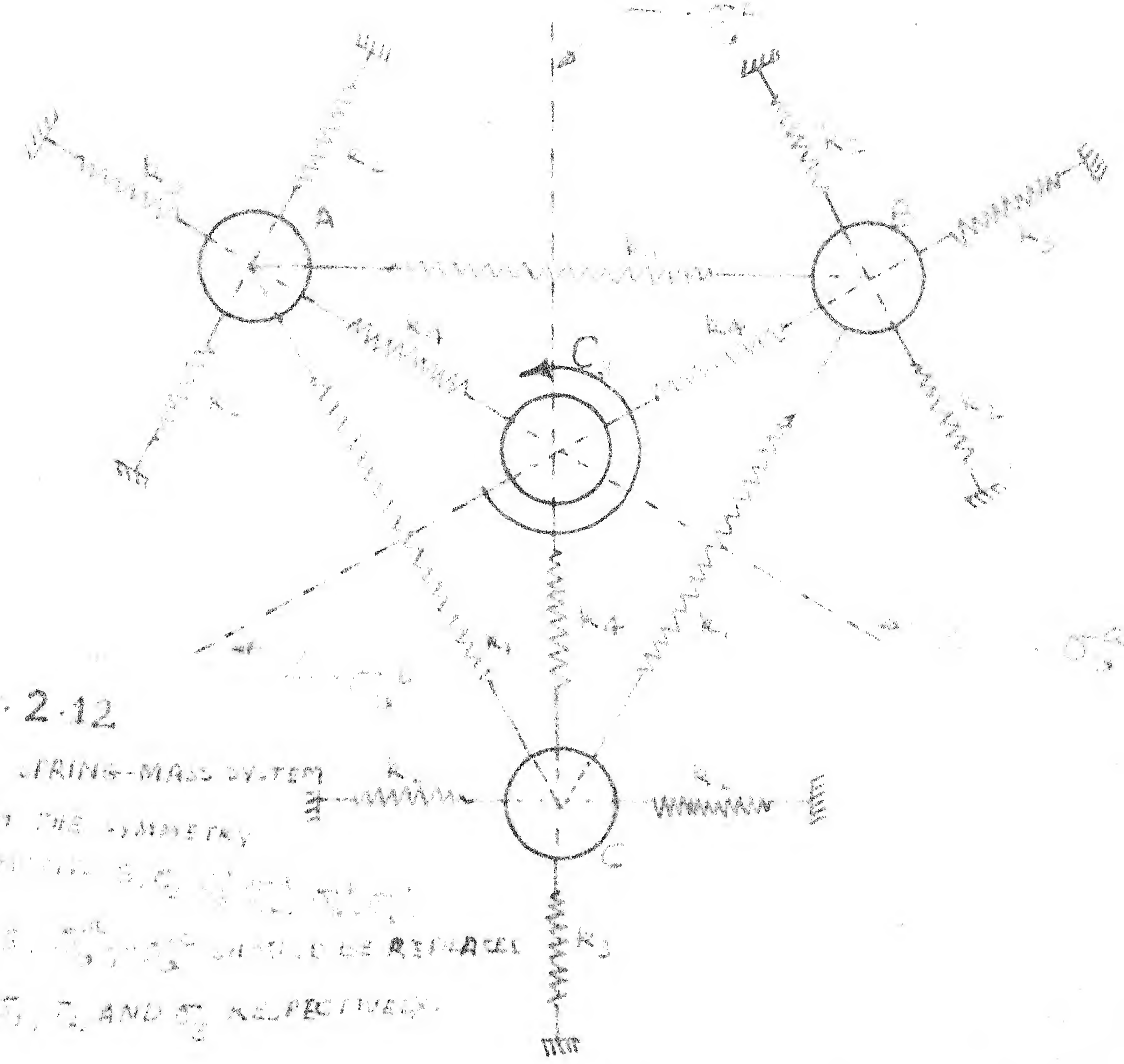


FIG. 2.12

THE SPRING-MASS SYSTEM WITH THE SYMMETRY ELEMENTS E, C_3, [illegible]

NOTE: [illegible] SHOULD BE REPLACED BY σ_1, σ_2 AND σ_3 RESPECTIVELY.

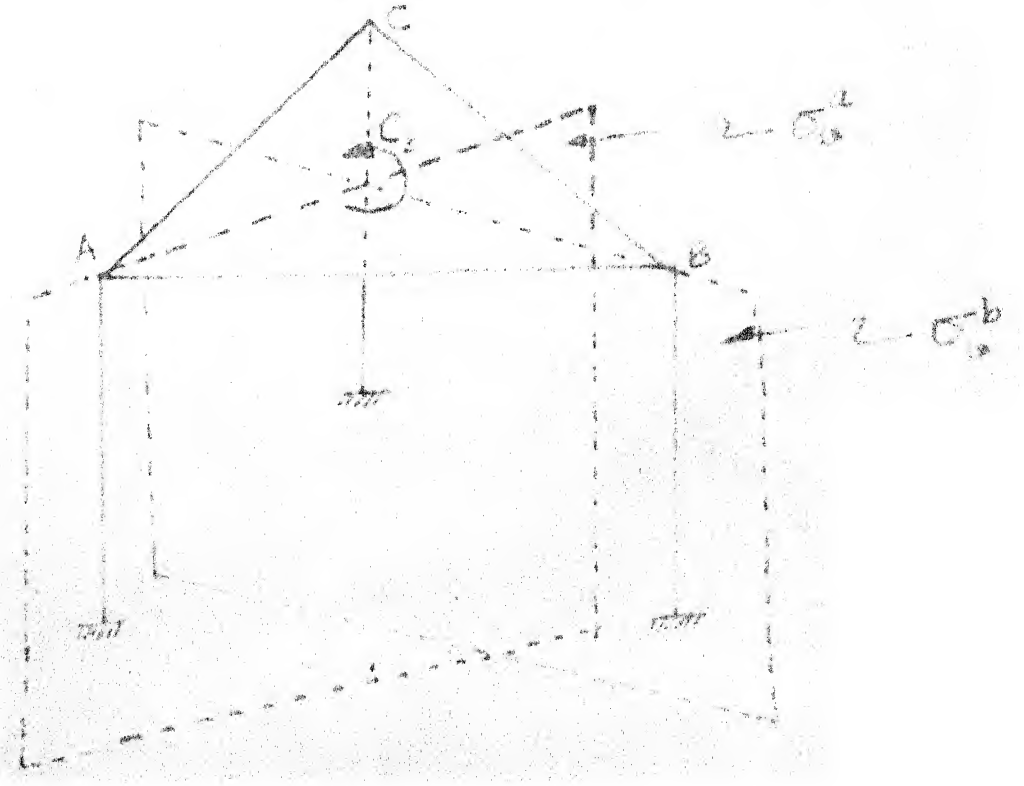


FIG. 2.13

THE TRIANGULAR FRAME WITH THE SYMMETRY ELEMENTS E, C_3, C_3^2, σ_v^a, σ_v^b, σ_v^c

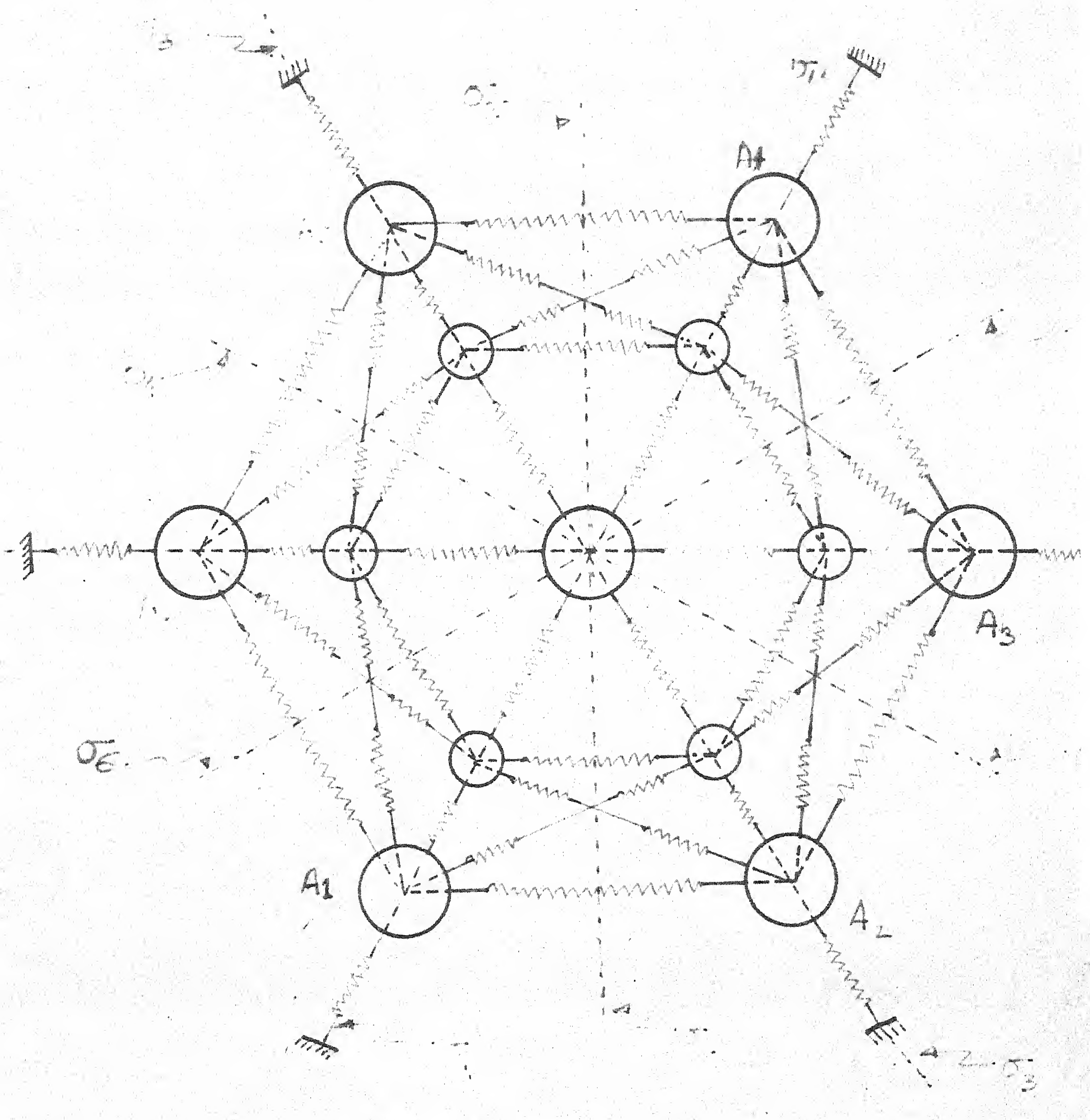


FIG. 2.15

SPRING MASS SYSTEM WITH SYMMETRY ELEMENTS $E, C_6, C_6^2, C_6^3, C_6^4, C_6^5, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6$.

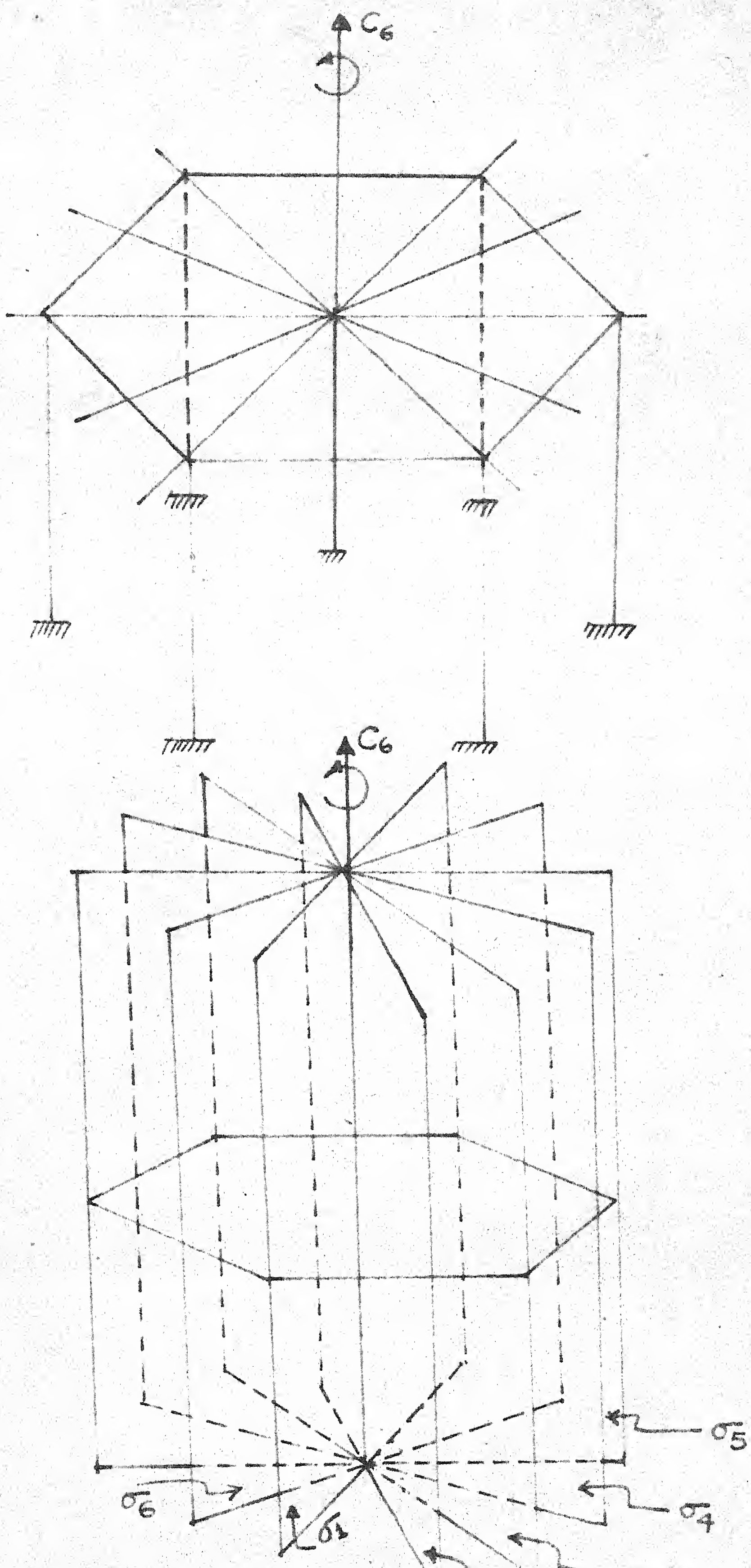


Fig. 2.15

HEXAGONAL SPACE FRAME WITH SYMMETRY ELEMENTS

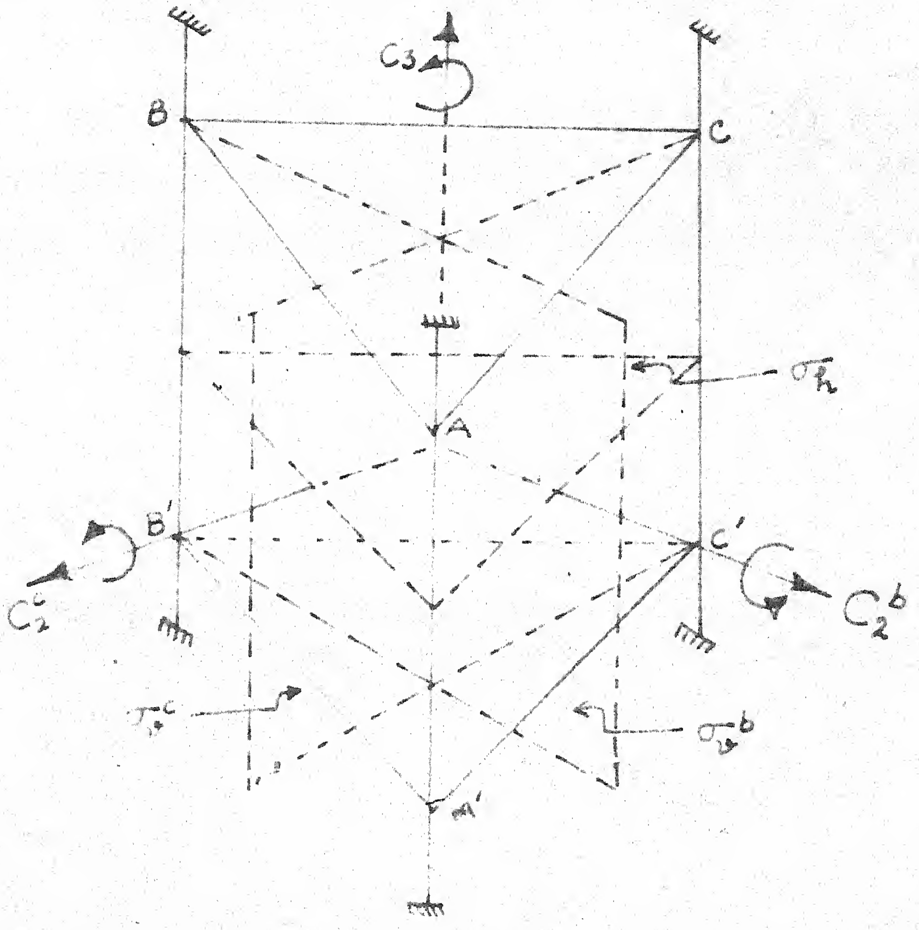


Fig. 2·16

FRAME WITH SYMMETRY ELEMENTS E, C_3, C_3^2, σ_v^a, σ_v^b, σ_v^c, σ_h, C_2^a, C_2^b, C_2^c, $S_3 = \sigma_h C_3$, $S_3^2 = \sigma_h C_3^2$

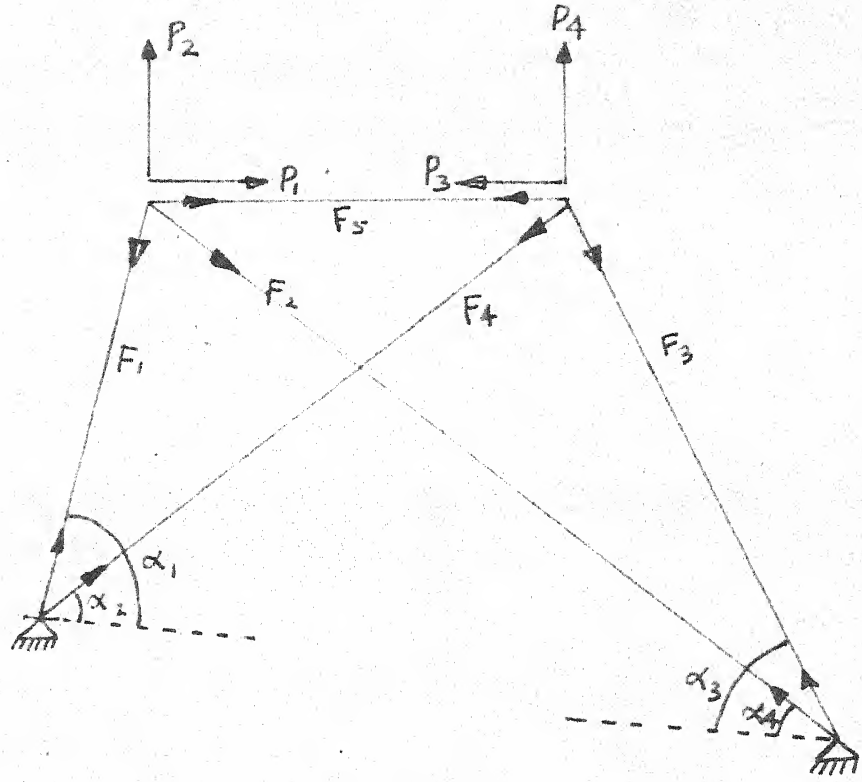


Fig. 2·19

GEOMETRICALLY UNSYMMETRIC BUT SYMMETRIC IN THE SENSE THAT ITS STIFFNESS MATRIX IS CENCTOSYMMETRIC.

THE MEMBER PARAMETERS E_i, A_i, L_i HAVE SIMILAR SUFFICES ($i=1,$... AS THE MEMBER FORCES F_i HAVE.

CHAPTER 3

A SEMI-INTUITIVE APPROACH TO SYMMETRY (THE MATRIX METHOD)

In this chapter the symmetric structural problems will be simplified by using certain transformations applied to "Stiffness Method". The method can as well be applied to "Flexibility Method". The transformations determined are based partly on intuitions and partly on analytical arguments and this is why the method is called semi-intuitive method. Only parts of complete symmetries of structures will be used. The complete symmetries may reduce the problem still more, but this aspect will not be discussed in detail in this chapter the reason of which will become clear as the method progresses. The short-comings and inadequacies of the method will also become apparent in the development of the method. The notations and terminology of previous chapter will be used.

The structural systems are assumed to have one or more of the following symmetries:

1. Single reflection plane
2. Double reflection planes orthogonal to each other
3. Triple reflection planes orthogonal to each other
4. Inversion symmetry
5. Cyclic symmetries

Here it is assumed that none of the reflection planes or axis of symmetry contain the "nodes" of the structural system. This assumption is essential due to the weakness of the intuitive approach.

Examples of these symmetrіes are given in Figs. 3.1(a),(b),(c),(d), 3.2(a), 3.2(b) and 3.2(c).

In what follows, the symmetries of load systems if at all there is any will not be taken into account explicitly except in buckling problems where atleast a part of the load system affects the stiffness matrix itself and need to have similar symmetries as structural system

3.1 SINGLE REFLECTION PLANE:

Consider a structural system having a reflection

plane σ (Fig. 3.3). Thus the plane σ divides the system in two regions R_1 and R_2. R_1 is mirror image of R_2 and vice-versa. Thus the symmetry elements of the system are:

E and σ.

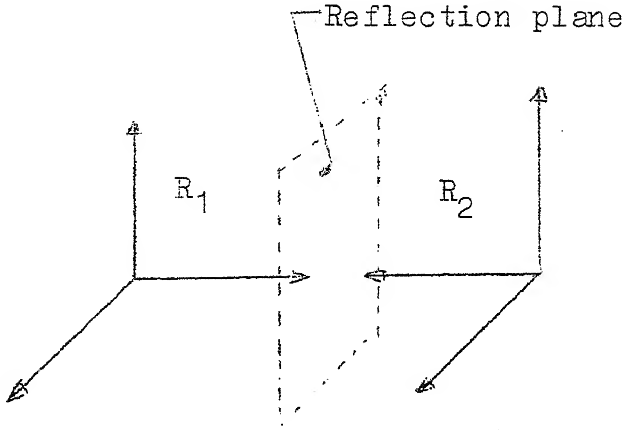


Fig. 3.3

Now if the global co-ordinate system for the two regions are mirror image of one another, then the "description" of the region R_1 w.r.t. its global co-ordinate system is exactly the same as that of R_2 w.r.t. its own global co-ordinate system.

Let there be n-degrees of freedoms in each region. Then there are 2n-degrees of freedom in the whole system

Now let

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}_1 \qquad F_1 = \begin{bmatrix} f_1 \\ f_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ f_n \end{bmatrix}_1$$

and force vector respectively in region R_1. The corresponding vectors for region R_2 are:

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}_2 \qquad F_2 = \begin{bmatrix} f_1 \\ f_2 \\ \cdot \\ \cdot \\ \cdot \\ f_n \end{bmatrix}_2$$

Displacement and force vectors for the whole structural system are:

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

Then the stiffness equation is

$$KX = F \qquad \ldots \qquad \ldots \qquad (1)$$

or

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{12}^T & K_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \quad \text{where } K = K^T$$

Now consider an operator T such that:

$$T \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_2 \\ X_1 \end{bmatrix}$$

$$T \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} F_2 \\ F_1 \end{bmatrix}$$

A matrix representation of T can be easily found to be:

$$T = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

because $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_2 \\ X_1 \end{bmatrix}$ etc.

Also one can see that $T = T^T$ and

$$T^2 = I$$

$$T^{-1} = T \quad \text{i e} \quad \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \qquad (2)$$

Thus T is a reflection operator whose matrix representation is an Involutory matrix. (Involutary matrix is one whose square euqls Identity matrix). This property reflects the reflection operation, viz. two consecutive reflection equals identity transformation.

Applying the operator T to the equation (1) one gets

$$TKX = TF$$

or $$TKT^{-1}TX = TF$$

or $$K'X' = F' \qquad \ldots \qquad \ldots \qquad (3)$$

where

$$X' = TX$$
$$F' = TF$$
$$K' = TKT^{-1}$$

Since operation of T is equivalent to interchanges of regions R_1 and R_2 and since R_1 and R_2 are exactly similar to each other w.r.t. their respective global co-ordinate systems. Therefore eq. (3) and eq. (1) are one and the same except for the interchanges of variables.

Therefore, $K' = K$

i.e. $$TKT^{-1} = K \qquad \ldots \qquad \ldots \qquad (4)$$

From Eqs. (2) $T^{-1} = T$

therefore, $$TKT^{-1} = TKT = K \qquad \ldots \qquad \ldots \qquad (5)$$

Also note that eqs. (4) means something more

i.e. $TK = KT$

or $\left[T, K\right] = 0$

T commutes with "K". From this property of K one can immediately draw some very important conclusions using a famous result for commuting operators (37), viz.

"If two operators commute, one can find a state function which is simultaneously eigen function of both operator". (which can be seen as follows: $(TK - KT)\psi = 0$ is satisfied if ψ is eigen function of T and K).

Thus finding of eigen functions of K reduces to that of finding eigen functions of T.

If ψ is eigen function of K i.e.

$$K\psi = k\psi \qquad \ldots \quad \ldots \qquad (6)$$

Then ψ is also eigen function of T and vice-versa.

i.e.

$$T\psi = t\psi \qquad \ldots \quad \ldots \qquad (7)$$

where k = eigen value of K

t = eigen value of T

From eqs. (7) one has

$$T^2 \psi = t\, T \psi$$

and from eqs. (2) and (6) again

$$I \psi = t \cdot t \psi$$

or $$t^2 = 1 \quad \text{or} \quad t = \pm 1$$

Therefore, eigen values of T are ± 1

i.e. $T\psi = t\psi$ breaks into two sets of eqs.

$$(T \pm I)\psi = 0$$

or $$\begin{bmatrix} \pm I & I \\ I & \pm I \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0$$

or $$X_1 \pm X_2 = 0$$

or eigen functions of T are

$$\psi_+ = \begin{bmatrix} X_1 \\ -X_1 \end{bmatrix}$$

$$\psi_- = \begin{bmatrix} X_1 \\ +X_1 \end{bmatrix}$$

So eigen functions of K are also given by

$$\psi_+ = \begin{bmatrix} X_1 \\ -X_1 \end{bmatrix}$$

$$\psi_- = \begin{bmatrix} X_1 \\ X_1 \end{bmatrix}$$

This naturally shows the symmetry of mode shapes i.e. ψ_+ is symmetric mode while ψ_- is antisymmetric mode. However the aim is not to show only this much but some-thing more than this.

Again considering the equation (5) and using the representation of T one gets.

$$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} K_{11} & K_{12} \\ K_{12}^T & K_{12} \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{12}^T & K_{22} \end{bmatrix}$$

OR

$$K_{11} = K_{22} = A \text{ (say)}$$

$$K_{12}^T = K_{12} = B \text{ (say)}$$

Therefore,

$$K = \begin{bmatrix} A & B \\ B & A \end{bmatrix}$$

So eq(1) has the form

$$\begin{bmatrix} A & B \\ B & A \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \quad ------ (8)$$

Now it has been shown that eigen functions of T and K are same and eigen function of

T are $\begin{bmatrix} X_1 \\ X_1 \end{bmatrix}$ and $\begin{bmatrix} X_1 \\ -X_1 \end{bmatrix}$ and are orthogonal to each other.

Noting that $\begin{bmatrix} X_1 \\ X_1 \end{bmatrix}$ etc. are set of n-vectors one can form a matrix by using these vectors as columns and thus let 'T_1' be the matrix so obtained then one of the possible choice of the T_1 is

$$T_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ I & -I \end{bmatrix}$$

where the factor $\frac{1}{\sqrt{2}}$ is chosen so that the square of a vector is equal to 1.

Since every column of T_1 is orthogonal to other columns. Therefore T_1 is orthogonal matrix.

Now using a well known theorem of matrix theory (38):

$T_1 \; T \; T_1^T$ = a diagonal matrix with the elements as eigen values of the matrix T, and $T_1KT_1^T$ = atleast a block diagonal matrix whose blocks are that giving the eigen-values of K.

Now, $KX = F$ can be written as

$$T_1KX = T_1F$$

or $T_1K(T_1^T)^{-1} \; T_1^TX = T_1F$

But $T_1^T = T_1$ and $T_1T_1^T = I$ (9)

Therefore $T_1KT_1T_1X = T_1F$

or $K^1 \; X^1 = F^1$ ... ... (10)

where $X^1 = T_1X = \frac{1}{\sqrt{2}} \begin{bmatrix} X_1 + X_2 \\ X_1 - X_2 \end{bmatrix}$

$$F^1 = T_1F = \frac{1}{\sqrt{2}} \begin{bmatrix} F_1 + F_2 \\ F_1 - F_2 \end{bmatrix}$$

$$K^1 = T_1KT_1$$

$$= \frac{1}{2} \begin{bmatrix} I & I \\ I & -I \end{bmatrix} \begin{bmatrix} A & B \\ B & A \end{bmatrix} \begin{bmatrix} I & I \\ I & -I \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} I & I \\ I & -I \end{bmatrix} \begin{bmatrix} A + B \\ A + B \end{bmatrix} \begin{bmatrix} A & -B \\ B & -A \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 2(A+B) & 0 \\ 0 & 2(A-B) \end{bmatrix} = \begin{bmatrix} A+B & 0 \\ 0 & A-B \end{bmatrix}$$

Eq. (10) reduces to:

$$\begin{bmatrix} A+B & 0 \\ 0 & A-B \end{bmatrix} \begin{bmatrix} X_1^1 \\ X_2^1 \end{bmatrix} = \begin{bmatrix} F_1^1 \\ F_2^1 \end{bmatrix}$$

or $$(A+B) \; (X_1+X_2) = (F_1 + F_2) \qquad \ldots (11)$$

$$(A-B) \; (X_1-X_2) = (F_1-F_2) \qquad \ldots (12)$$

Now solution of the actual problem which is a 2n set of simultaneous equations reduces to 2-sets of n-simultaneous equations which simplifies the problems quite a lot.

It can be seen that the equations (11) and (12) correspond to what are called anti-symmetric and symmetric parts of a mirror symmetric system (32). To get these forms of equations in ordinary structural analysis one puts a lot of effort of intuitions in breaking the solutions in symmetric and antisymmetric form.

Now solving for X_1 and X_2 one can easily get X_1, X_2 by the definition of X_1^1 and X_2^1 viz.

$$\begin{bmatrix} X_1^1 \\ X_2^1 \end{bmatrix} = T_1 \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

or

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = T_1 \begin{bmatrix} X_1^1 \\ X_2^1 \end{bmatrix} \qquad (\text{Because } T_1 T_1 = I)$$

$$= \frac{1}{2} \begin{bmatrix} X_1^1 + X_2^1 \\ X_1^1 - X_2^1 \end{bmatrix}$$

With these discussions one can now conclude the following theorems:

<u>THEOREM 1:</u>

Given a structural system with a plane of symmetry; the stiffness matrix can be written in the form:

$$K = \begin{bmatrix} A & B \\ B & A \end{bmatrix}$$ w.r.t. the two mirror image global co-ordinate systems

for the regions one left and the other right to the plane of symmetry.

<u>THEOREM 2:</u>

The problem of structural analysis of a structural system with a plane of symmetry breaks into two problems

of order half to that of total problem i.e.

$$\left[K\right]\begin{bmatrix}X_1\\X_2\end{bmatrix}=\begin{bmatrix}F_1\\F_2\end{bmatrix} \text{ goes to } \begin{array}{l}(A+B)\ (X_1+X_2)\ = F_1+F_2\\(A-B)\ (X_1-X_2)\ = F_1-F_2\end{array}$$

Following corollaries may be derived very easily.

Corollary 1:

If along with the structural system the load system is also having a plane of symmetry, then the displacement is also symmetric. If however load system is antisymmetric w.r.t. the plane of symmetry of structure, then the displacement will also be anti-symmetric w.r.t. the plane of symmetry. In any of the above case one needs to solve for one side of the plane of symmetry. This concides Broek's result (32).

Proof:

(i) Load is anti-symmetric i.e. $F_1 = -F_2$

Then $F_1 + F_2 = 0$ and $F_1 - F_2 = 2F_1$ and therefore the eqs. (11) and (12) reduce to:

$$(A + B)\ (X_1 + X_2) = 0$$

$$(A - B)\ (X_1 - X_2) = 2F_1$$

Since $A + B$ = non-singular matrix

Therefore $X_1 + X_2 = 0$

i.e. $X_2 = -X_1$

One gets

$$(A - B) \quad X_1 = F_1$$

$$X_2 = - X_1$$

(ii) Load is symmetric i.e. $F_1 = F_2$

$$F_1 - F_2 = 0 \qquad F_1 + F_2 = 2F_1$$

From $(A-B)(X_1 - X_2) = F_1 - F_2 = 0$ one gets

$$X_1 - X_2 = 0$$

i.e. $X_1 = X_2$

and the total problem reduces to

$$(A + B) X_1 = F_1$$

$$X_2 = X_1$$

Corollary 2

The characteristic equation determining the natural frequencies for the total problem for a strutural system with a plane of symmetry breaks into

two characteristic equations of order half to that of total system.

Proof:

Let M = mass matrix of the system

K = stiffness matrix

Because of plane of symmetry, M will have at least the symmetry of the K. (If one applies Archer's (39) procedure to determine a consistent mass matrix and of-course for lumped mass system), i.e. M can also be written as

$$M = \begin{bmatrix} C & D \\ D & C \end{bmatrix}$$

Eqs. of motion is

$$KX + M\ddot{X} = 0$$

or

$$(K - w^2M)X = 0$$

$$\det \left| K-w^2 M \right| = 0$$ gives the frequencies of the system.

or

$$\det \left| \begin{bmatrix} A & B \\ B & A \end{bmatrix} - w^2 \begin{bmatrix} C & D \\ D & C \end{bmatrix} \right| = 0$$ gives the frequencies.

or

$$\det \left| \begin{bmatrix} A+B & 0 \\ 0 & A-B \end{bmatrix} - w^2 \begin{bmatrix} C+D & 0 \\ 0 & C-D \end{bmatrix} \right| = 0$$

or

$$\det \left| A+B - w^2(C+D) \right| \det \left| A-B-w^2(C-D) \right| = 0$$

or

$$\det \left| A+B - w^2 C - w^2 D \right| = 0 \quad \cdots\cdots (13)$$

$$\det \left| A-B - w^2 C + w^2 D \right| = 0 \quad \cdots\cdots (14)$$

These two eqs. give the frequencies.

Corollary - 3:

For a structural system with a plane of symmetry, the buckling load for the system under the axial loads with same symmetry will also be determined by Eqs. of the form (13) and (14) because the buckling loads are determined by $\det|K| = 0$ (40).

i.e. $\det|K| = 0$ reduces to

$$\det|A+B| = 0$$

$$\det|A-B| = 0$$

where A and B are now also functions of axial loads. This gives a proof to the one of the statements given by Salem (23).

Proof:

The proof of this corollary is exactly similar to that corollary 2.

To get a feel of what has been done, this procedure is applied toafew examples.

EXAMPLE 1

Vibration of 6-masses and springs. Consider the following system:

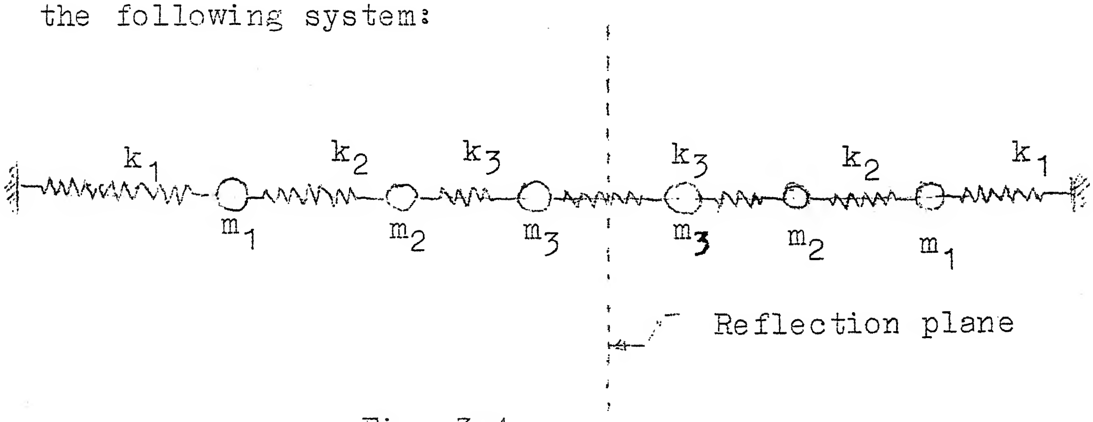


Fig. 3.4
THE SPRING-MASS SYSTEM

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_1 ; \quad X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_2 , \quad X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

Eqs. of motion is

$$M\ddot{X} + KX = F(t)$$

where $M = \text{diag.}\ (m_1, m_2, m_3, m_1, m_2, m_3) = \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix}$

$$K = \begin{bmatrix} A & B \\ B & A \end{bmatrix}$$

where $A = \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3+k_4 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -k_4 \end{bmatrix}$, $C = \begin{bmatrix} m_1 & & 0 \\ & m_2 & \\ 0 & & m_3 \end{bmatrix}$

From corollary - 2, the natural frequencies are given by:

$$\det \left| A+B - w^2 C \right| = 0 \quad \text{and} \quad \det \left| A-B - w^2 C \right| = 0$$

or

$$\begin{vmatrix} k_1+k_2-m_1w^2 & -k_2 & 0 \\ -k_2 & k_2+k_3-m_2w^2 & -k_3 \\ 0 & -k_3 & k_3 \text{ [redacted] } -m_3w^2 \end{vmatrix} = 0$$

and

$$\begin{vmatrix} k_1+k_2-m_1w^2 & -k_2 & 0 \\ -k_2 & k_2+k_3-m_2w^2 & -k_3 \\ 0 & -k_3 & k_3+2k_4-m_3w^2 \end{vmatrix} = 0$$

With $k_1 = k_2 = k_3 = k_4 = 1 = m_1 = m_2 = m_3$,

One gets following values of w^2:

Corresponding to symmetric modes

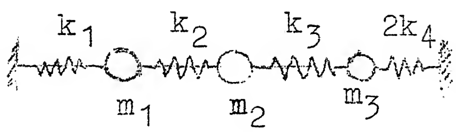
$$w_1^2 = 0.198$$
$$w_2^2 = 1.570$$
$$w_3^2 = 3.22$$

Corresponding to antisymmetric modes

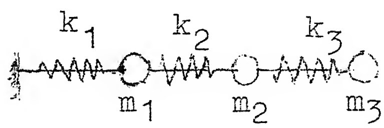
$$w_4^2 = 0.9$$
$$w_5^2 = 2.5$$
$$w_6^2 = 3.49$$

OBSERVATIONS:

One can see that the matrices A+B and A-B correspond to the stiffness matrices of the following sub-systems of the complete system.

I : 

Corresponding to stiffness matrix A-B

II : 

Corresponding to stiffness matrix A+B

Thus the complete problem with a reflection plane of symmetry breaks into two smaller sub-problems. Also, that one needs to solve only the sub-problem I and to get the solution for II one needs to set $k_4 = 0$

in the sub-problem I. Physically it means that out of 6 modes of the system, 3-modes will be those in which middle spring will play no role and the rest 3-modes will be ones for which the middle spring produces effects of that of its double. Putting in another term, sub-problem II is that in which all masses of R_1 move exactly similar to those of R_2 and sub-problem I is that in which all masses of R_1 move exactly opposite to those of R_2 corresponding to the so called anti-symmetric and symmetric modes respectively.

Also if one is interested only in the lowest natural frequency of the system, one needs only to solve the sub-problem II for its lowest frequency not the whole problem and therefore if one uses the power method, one needs the mutliplications by 3×3 matrices not the 6×6 complete matrices and also one needs to invert only 3×3 matrix $A+B$ not the 6×6 K matrix. To get a feel for the saving of the effort in computing the lowest frequency and corresponding mode by power method a 2n-masses system with a plane of symmetry is considered.

$$\text{Total stiffness matrix} = 2n \times 2n$$

A+B = n x n matrix with same band width as K

No. of multiplication in inversion of an nxn matrix is $\sim n^3$

No. of multiplication in inversion of an 2nx2n is $\sim 8n^3$

Reduction in number of multiplication in inversion by considering symmetry by this method is $\sim 7n^3$.

Now if the problem is that of 200 masses with a reflection plane. Then the number of reductions in multiplications $\sim 7 \times 10^6$.

Similarly in iteration process also, the number of reductions in multiplications in m-iteration is $\sim 7n^2m$.

If $m = 10$

$n = 100$

then no. of reduction in multiplication $\approx 7 \times 10^5$.

EXAMPLE - 2

Consider the truss shown in Fig. (3.5) shown in the n page. This truss was considered in Chapter 2 in connection with showing the role of geometry and members properties in symmetry of the structure. Here and in all the work as has been pointed out in the

end of II Chapter both geometrical and members are assumed to contribute equally to the resultant symmetry of structure. The structure has two symmetry elements (E, σ). The co-ordinates for the two regions are chosen as were required in the general discussion.

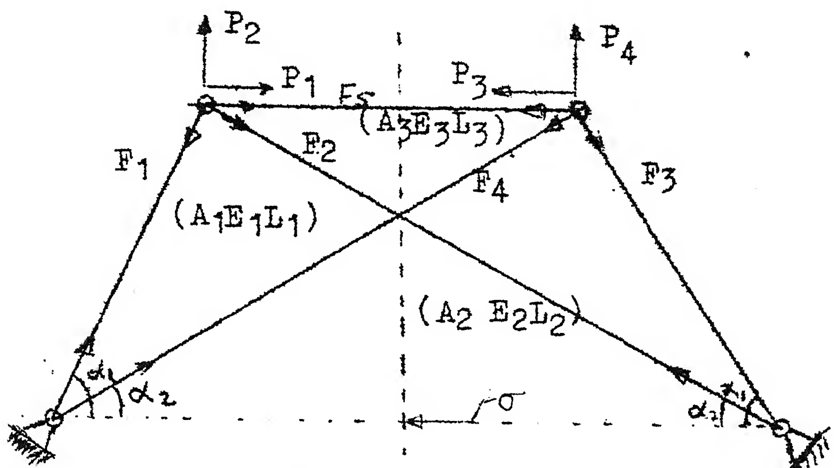


Fig. 3.5
THE SYMMETRIC TRUSS

The stiffness equation is,

$$KX = P$$

and

$$K = \bar{A}S\bar{A}^T$$

$$\bar{A} = \left[\begin{array}{cc|cc|c} \cos\alpha_1 & -\cos\alpha_2 & 0 & & -1 \\ \sin\alpha_1 & \sin\alpha_2 & & & 0 \\ \hline & & \cos\alpha_1 & -\cos\alpha_2 & -1 \\ 0 & & \sin\alpha_1 & \sin\alpha_2 & 0 \end{array}\right] = \left[\begin{array}{c|c|c} \bar{A}_1 & 0 & a \\ \hline 0 & \bar{A}_1 & a \end{array}\right]$$

This has been taken from Chapter 2 with some modifications.

$$S = \left[\begin{array}{cc:cc:c} \frac{A_1E_1}{L_1} & & & & \\ & \frac{A_2E_2}{L_2} & & 0 & \\ \hdashline & & \frac{A_1E_1}{L_1} & & 0 \\ & 0 & & \frac{A_2E_2}{L_2} & \\ \hdashline & & 0 & & \frac{A_3E_3}{L_3} \end{array}\right] = \left[\begin{array}{c:c:c} B_1 & 0 & 0 \\ \hdashline 0 & B_1 & 0 \\ \hdashline 0 & 0 & b \end{array}\right]$$

$$\text{where } B_1 = \begin{bmatrix} \frac{A_1E_1}{L_1} & 0 \\ 0 & \frac{A_2E_2}{L_2} \end{bmatrix}, \quad b = \frac{A_3E_3}{L_3}$$

$$\text{So, } K = \bar{A}S\bar{A}^T = \begin{bmatrix} \bar{A}_1B_1\bar{A}_1^T + baa^T & baa^T \\ baa^T & \bar{A}_1B_1\bar{A}_1^T + baa^T \end{bmatrix} = \begin{bmatrix} A & B \\ B & A \end{bmatrix}$$

Total stiffness matrix is given in the Table (3.1). Thus applying the result derived earlier (Theorem 2), one gets:

$$(A+B)\ (X_1 + X_2) = P_1 + P_2$$

$$(A-B)\ (X_1 - X_2) = P_1 - P_2$$

TABLE 3.1

STIFFNESS MATRIX FOR THE TRUSS SHOWN IN FIG.3.5

	1	2
1	$\frac{A_3E_3}{L_3} + \frac{A_1E_1}{L_1}\mathrm{Cos}^2\alpha_1 + \frac{A_2E_2}{L_2}\mathrm{Cos}^2\alpha_2$	$\frac{A_1E_1}{L_1}\mathrm{Cos}\alpha_1\mathrm{Sin}\alpha_1 - \frac{A_2E_2}{L_2}\mathrm{Cos}\alpha_2\mathrm{Si}$
2	$\frac{A_1E_1}{L_1}\mathrm{Cos}\alpha_1\mathrm{Sin}\alpha_1 - \frac{A_2E_2}{L_2}\mathrm{Cos}\alpha_2\mathrm{Sin}\alpha_2$	$\frac{A_1E_1}{L_1}\mathrm{Sin}^2\alpha_1 + \frac{A_2E_2}{L_2}\mathrm{Sin}^2\alpha_2$
3	$\frac{A_3E_3}{L_3}$	0
4	0	0

	3	4
1	$\frac{A_3E_3}{L_3}$	0
2	0	0
3	$\frac{A_3E_3}{L_3} + \frac{A_1E_1}{L_1}\mathrm{Cos}^2\alpha_1 + \frac{A_2E_2}{L_2}\mathrm{Cos}^2\alpha_2$	$\frac{A_1E_1}{L_1}\mathrm{Cos}\alpha_1\mathrm{Sin}\alpha_1 - \frac{A_2E_2}{L_2}\mathrm{Cos}\alpha_2\mathrm{S}$
4	$\frac{A_1E_1}{L_1}\mathrm{Cos}\alpha_1\mathrm{Sin}\alpha_1 - \frac{A_2E_2}{L_2}\mathrm{Cos}\alpha_2\mathrm{Sin}\alpha_2$	$\frac{A_1E_1}{L_1}\mathrm{Sin}^2\alpha_1 + \frac{A_2E_2}{L_2}\mathrm{Sin}^2\alpha_2$

$$X_1 + X_2 = (A+B)^{-1} \; (P_1 + P_2)$$

$$X_1 - X_2 = (A-B)^{-1} \; (P_1 - P_2)$$

$$X_1 = \frac{1}{2}\left[(A+B)^{-1} \; (P_1+P_2) + (A-B)^{-1} \; (P_1-P_2) \right]$$

$$X_2 = \frac{1}{2}\left[(A+B)^{-1} \; (P_1+P_2) - (A-B) \; (P_1-P_2) \right]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ - \\ x_3 \\ x_4 \end{bmatrix} = \left[\begin{array}{cccc} a_1\left(\frac{1}{D_1}+\frac{1}{D_2}\right) & a_2\left(\frac{1}{D_1}+\frac{1}{D_2}\right) & a_1\left(\frac{1}{D_1}-\frac{1}{D_2}\right) & a_2\left(\frac{1}{D_1}-\frac{1}{D_2}\right) \\ a_2\left(\frac{1}{D_1}+\frac{1}{D_2}\right) & \left(\frac{b_1}{D_1}+\frac{b_2}{D_2}\right) & a_2\left(\frac{1}{D_1}-\frac{1}{D_2}\right) & \frac{b_1}{D_1}-\frac{b_2}{D_2} \\ \hdashline a_1\left(\frac{1}{D_1}-\frac{1}{D_2}\right) & a_2\left(\frac{1}{D_1}-\frac{1}{D_2}\right) & a_1\left(\frac{1}{D_1}+\frac{1}{D_2}\right) & a_2\left(\frac{1}{D_1}+\frac{1}{D_2}\right) \\ a_2\left(\frac{1}{D_1}-\frac{1}{D_2}\right) & \frac{b_1}{D_1}-\frac{b_2}{D_2} & a_2\left(\frac{1}{D_1}+\frac{1}{D_2}\right) & \frac{b_1}{D_1}+\frac{b_2}{D_2} \end{array}\right] \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}$$

where $a_1 = \frac{A_1 E_1}{L_1} \sin^2\alpha_1 + \frac{A_2 E_2}{L_2} \sin^2\alpha_2$

$$a_2 = \frac{A_2 E_2}{L_2} \cos\alpha_2 \sin\alpha_2 - \frac{A_1 E_1}{L_1} \cos\alpha_1 \sin\alpha_1$$

$$b_1 = \frac{2E_3 A_3}{L_3} + b_2$$

$$b_2 = \frac{A_1 E_1}{L_1} \cos^2\alpha_1 + \frac{A_2 E_2}{L_2} \cos^2\alpha_2$$

and

$$D_2 = \frac{A_1 A_2 E_1 E_2}{L_1 L_2} \operatorname{Sin}^2 (\alpha_1 - \alpha_2)$$

$$D_1 = \frac{2A_3 E_3}{L_3} a_1 + D_2$$

Now one can find the member forces by,

$$F = Se = S\bar{A}^T X$$

$$= \left[\begin{array}{cc:cc} \frac{A_1E_1}{L_1}\operatorname{Cos}\alpha_1 & \frac{A_1E_1}{L_1}\operatorname{Sin}\alpha_1 & \multicolumn{2}{c}{0} \\ -\frac{A_2E_2}{L_2}\operatorname{Cos}\alpha_2 & \frac{A_2E_2}{L_2}\operatorname{Sin}\alpha_2 & & \\ \hdashline & & \frac{A_1E_1}{L_1}\operatorname{Cos}\alpha_1 & \frac{A_1E_1}{L_1}\operatorname{Sin}\alpha_1 \\ \multicolumn{2}{c:}{0} & -\frac{A_2E_2}{L_2}\operatorname{Cos}\alpha_2 & \frac{A_2E_2}{L_2}\operatorname{Sin}\alpha_2 \\ \hdashline \frac{A_3E_3}{L_3} & 0 & -\frac{A_3E_3}{L_3} & 0 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} \frac{A_1E_1}{L_1}\operatorname{Cos}\alpha_1 & \frac{A_1E_1}{L_1}\operatorname{Sin}\alpha_1 \\ -\frac{A_2E_2}{L_2}\operatorname{Cos}\alpha_2 & \frac{A_2E_2}{L_2}\operatorname{Sin}\alpha_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} \frac{A_1E_1}{L_1} \operatorname{Cos}\alpha_1 & \frac{A_1E_1}{L_1} \operatorname{Sin}\alpha_1 \\ -\frac{A_2E_2}{L_2} \operatorname{Cos}\alpha_2 & \frac{A_2E_2}{L_2} \operatorname{Sin}\alpha_2 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$$

$$f_s = - \frac{A_3E_3}{L_3} (x_1 + x_3)$$

By substituting the values of x_1, x_2, x_3 and x_4 the mamber forces are at once found in terms of joint forces p_1, p_2, p_3 and p_4.

It can be seen that the symmetry has helped calculations not only in inverting the stiffness matrix but also in computing member forces etc. Once the symmetry of structural system is inflicted in to that of of its stiffness matrix etc., a tremendous labour is saved.

One can see how difficult it would have been to solve this example in such a general form had this procedure not been used. The Broek's (32) **intuitive** rules also would have been used to simplify this problem but they would not tell any way to proceed to make use of higher symmetries, while the present procedure indicates a way to make use of higher symmetries on similar lines.

3.2 DOUBLE REFLECTION SYMMETRY:

Here the structural systems have got two orthogonal planes of symmetry. As shown in Figure (3.6), the two orthogonal planes divide the structural systems in 4-regions which are mirror images of each other w.r.t. these planes. Now if the global co-ordinate systems for the 4-regions are so chosen that they themselves are mirror images of each other w.r.t the reflection planes, then the description of any one region w.r.t. its own global co-ordinate system is exactly similar to that of any other region w.r.t. the corresponding global co-ordinate system. To fix up the idea the reflection planes are named σ_1 and σ_2 and the 4 - regions obtained from division by σ_1 and σ_2 are named R_1, R_2, R_3 and R_4 as shown in Fig. (3.6). Symmetry elements of the system are (E, σ_1, σ_2 C_2). However only σ_1 and σ_2 will be used.

Let the system has 4n-degrees of freedom, n-degrees of freedom corresponding to each region.

Let X_1, X_2, X_3 and X_4 and F_1, F_2, F_3 and F_4 are the displacement and force vectors respectively for the respective regions R_1, R_2, R_3 and R_4 w.r.t. the

Stiffness eqs. is

$$KX = F \qquad \ldots\ldots\ldots \quad (1)$$

Now consider the following operators which correspond to interchanges of the regions R_1, R_2, R_3 and R_4.

$$T_1 \cdot \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} X_3 \\ X_4 \\ X_1 \\ X_2 \end{bmatrix} \quad \ldots \text{ Corresponding to operation } \sigma_1$$

$$T_2 \cdot \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} X_2 \\ X_1 \\ X_4 \\ X_3 \end{bmatrix} \quad \ldots \text{ Corresponding to operation } \sigma_2$$

Since under reflections w.r.t. σ_1 and σ_2 the physics of the problem does not change,

therefore the problem does not change under T_1 and T_2 also i.e.

$$T_1KX = T_1F \qquad \text{------------} \quad (2)$$

$$T_2KX = T_2F \qquad \ldots\ldots\ldots\ldots \quad (3)$$

are exactly similar to (except for the order of variables),

$$KX = F$$

Eqs. (2) and (3) can be written as

$$T_1KT_1^{-1}\ T_1X = T_1F \quad \text{or} \quad K'\ X' = F'$$

$$T_2KT_2^{-1}\ T_2X = T_2F \quad \text{or} \quad K''\ X'' = F''$$

where $T_1X = X'$, $T_2X = X''$, $T_1F = F_1'$ and $T_2F = F''$

and $T_1KT_1^{-1} = K'$ and $T_2KT_2^{-1} = K''$

Physics of the problem does not change under T_1 and T_2 simply means that K' and K'' are the same as K,

i.e. $$T_1KT_1^{-1} = K \qquad (4)$$

$$T_2KT_2^{-1} = K \qquad (5)$$

The matrix representations of T_1 and T_2 can be written as

$$T_1 = \begin{bmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{bmatrix}$$

where I is n x n unit matrix.

Now it can be easily seen that,

$T_1 T_1 = I$ and $T_2\, T_2 = I$ i.e. $T_1^{-1} = T_1$ and $T_2^{-1} = T_2 \ldots$

$$\ldots (6)$$

$$T_1^T = T_1 \text{ and } T_2^T = T\;, \quad T_1 T_2 = T_2 T_1 \qquad \ldots (7)$$

i.e. T_1 and T_2 are orthogonal involutary matrices which physically means that two operations of same reflection convert the system into itself. Therefore eqs. (4) and (5) can be written as,

$$T_1 K T_1 = K$$

$$T_2\, K T_2 = K$$

written explicitly these eqs. are,

$$\begin{bmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix} \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{12}^T & K_{22} & K_{23} & K_{24} \\ K_{13}^T & K_{23}^T & K_{33} & K_{34} \\ K_{14}^T & K_{24}^T & K_{34}^T & K_{44} \end{bmatrix} \begin{bmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix} = K$$

and,

$$\begin{bmatrix} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{bmatrix} \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{12}^T & K_{22} & K_{23} & K_{24} \\ K_{13}^T & K_{23}^T & K_{33} & K_{34} \\ K_{14}^T & K_{24}^T & K_{34}^T & K_{44} \end{bmatrix} \begin{bmatrix} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{bmatrix} = K$$

or

$$\begin{bmatrix} K_{33} & K_{34} & K_{13}^T & K_{23}^T \\ K_{34}^T & K_{44} & K_{14}^T & K_{24}^T \\ K_{13} & K_{14} & K_{11} & K_{12} \\ K_{23} & K_{24} & K_{12}^T & K_{22} \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{12}^T & K_{22} & K_{23} & K_{24} \\ K_{13}^T & K_{23}^T & K_{33} & K_{34} \\ K_{14}^T & K_{24}^T & K_{34}^T & K_{44} \end{bmatrix}$$

and

$$\begin{bmatrix} K_{22} & K_{12}^T & K_{24} & K_{23} \\ K_{12} & K_{11} & K_{14} & K_{13} \\ K_{24} & K_{14}^T & K_{44} & K_{24} \\ K_{23}^T & K_{13}^T & K_{34} & K_{33} \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{12}^T & K_{22} & K_{23} & K_{24} \\ K_{13}^T & K_{23}^T & K_{33} & K_{34} \\ K_{14}^T & K_{24}^T & K_{34}^T & K_{44} \end{bmatrix}$$

Thus

$$K = \begin{bmatrix} A & B & C & D \\ B & A & D & C \\ C & D & A & B \\ D & C & B & A \end{bmatrix} \quad \text{with } A = A^T,\ B=B^T,\ C = C^T \text{ and } D = D^T$$

where

$$A = K_{11} = K_{22} = K_{33} = K_{44}$$

$$B = K_{12} = K_{12}^T = K_{34} = K_{34}^T$$

$$C = K_{13} = K_{13}^T = K_{24}^T = K_{24}$$

$$D = K_{14} = K_{14}^T = K_{23} = K_{23}^T$$

Incidently one can see that if the physics of the problem is invariant under T_1 and T_2 individually, then it should remain invariant under $T_1 T_2$, T_2T_1, $T_1T_2T_1$ and $T_2T_1T_2$ etc. Thus the problem should remain invariant under the following set of different operations.

$$E,\ T_1,\ T_2 \ \text{ and } \ T_1 T_2 = T_2 T_1$$

This broad invariance of the problem then poses another question; whether all of the different

operations will play any role in the simplification of the problem? The answer will be cleared in the Vth Chapter where the concepts of group theory will be used. As far as the simplifications in stiffness matrix is concerned, the answer to above question is 'No' which can be seen as follows:

Given

$$T_1 KT_1 = K$$

$$T_2 KT_2 = K$$

$$T_2T_1KT_1T_2 = T_2T_1 K(T_2T_1)^{-1} = T_2KT_2 = K$$

where $T_1 = T_1^{-1}$, $T_2 = T_2^{-1}$ and $(T_2T_1)^{-1} = T_1^{-1} T_2^{-1}$

$= T_1 T_2$ has been used.

Similarly

$$T_1 T_2KT_2T_1 = T_1T_2K(T_1T_2)^{-1} = T_1KT_1 = K$$

and

$$T_2 T_1 T_2 KT_2 T_1 T_2 = T_2 KT_2 = K$$

$$T_1 T_2 T_1 KT_1 T_2 T_1 = T_1 KT_1 = K \text{ etc.}$$

Therefore $T_1KT_1 = K = T_2KT_2$ is implied by all other different operations.

Now if the eqs. (4) and (5) are written in a little different form as follows:

$$T_1K = KT_1$$

$$T_2K = KT_2$$

i.e. $$\left[T_1, K\right] = 0 \qquad \ldots \quad (8)$$

$$\left[T_2, K\right] = 0 \qquad \ldots \quad (9)$$

and also $\left[T_1, T_2\right] = 0$ from eq. (7).

where again by $\left[T_i, K\right]$ is meant the commutator of,

T_i and K etc.

Again using the famous theorem (37) of linear operators i.e. if a set of linear operators $L_1, L_2, \ldots L_n$ commute with each other and also with another linear operator L then one can find a function Ψ which is simultaneously eigen function of $L_1, L_2 \ldots\ldots L_n$ and L.

i.e.

$$L_i\Psi = l_i\Psi \qquad i = 1, 2, \ldots\ldots n$$

$$L\Psi = l\Psi$$

(where l_1 l_2 - - - l_n and l are eigenvalues of L_1, L_2 - - - L_n and L respectively).

i.e. eigen functions of L must have at least all those properties which L_1, L_2 - - - L_n's eigen functions have. In the present situation these linear operators are only two i.e. T_1 and T_2.

Let Ψ_1 and Ψ_2 be the eigen functions and t_1 and t_2 be eigen values of T_1 and T_2 respectively then

$$T_1 \Psi_1 = t_1 \Psi_1$$

$$T_2 \Psi_2 = t_2 \Psi_2$$

$$T_1^2 \Psi_1 = t_1 T_1 \Psi_1 = t_1^2 \Psi_1$$

or $$\Psi_1 = t_1^2 \Psi_1$$

or $$t_1^2 = 1 \text{ or } t_1 = \pm 1$$

and $$T_2^2 \Psi_2 = t_2 T_2 \Psi_2 = t_2^2 \Psi_2$$

or $$\Psi_2 = t_2^2 \Psi_2$$

or $$t_2^2 = 1 \quad \text{i.e. } t_2 = \pm 1$$

Thus $T_1 \Psi_1 = t_1 \Psi_1$ and $T_2 \Psi_2 = t_2 \Psi_2$ reduce to

$(T_1 \pm I)\Psi_1 = 0$ and $(T_2 \pm I)\Psi_2 = 0$ respectively.

Since T_1 and T_2 are $4n \times 4n$ matrices.

Therefore I is also a $4n \times 4n$ matrix and ψ_1 and ψ_2 are $4n \times 1$ vectors.

Let
$$\psi_1 = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} \text{ and } \psi_2 = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix}$$

Then the eigen value problems (10) reduces to:

$$\begin{bmatrix} \pm I & 0 & I & 0 \\ 0 & \pm I & 0 & I \\ I & 0 & \pm I & 0 \\ 0 & I & 0 & \pm I \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = 0 \text{ and } \begin{bmatrix} \pm I & 0 & I & 0 \\ I & \pm I & 0 & 0 \\ 0 & 0 & \pm I & I \\ 0 & I & 0 & \pm I \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix} = 0$$

or
$$X_1 = \pm X_2$$
$$X_3 = \pm X_4$$
$$Y_1 = Y_2 = Y_3 = Y_4$$
$$-Y_1 = Y_2 = Y_3 = Y_4$$

It can be easily seen that a suitable set of eigen-vectors for T_1 and T_2 are,

$$\frac{1}{2}\begin{bmatrix} I \\ I \\ I \\ I \end{bmatrix} \quad \frac{1}{2}\begin{bmatrix} I \\ -I \\ I \\ -I \end{bmatrix} \quad \frac{1}{2}\begin{bmatrix} I \\ I \\ -I \\ -I \end{bmatrix} \text{ and } \frac{1}{2}\begin{bmatrix} I \\ -I \\ -I \\ I \end{bmatrix}$$

The matrix obtained by putting these 4n-vectors as column is,

$$T_3 = \frac{1}{2}\begin{bmatrix} I & I & I & I \\ I & -I & I & -I \\ I & I & -I & -I \\ I & -I & -I & I \end{bmatrix}$$

It can be seen that $T_3^T = T_3$

Also $T_3\,T_3 = I$ i.e. $T_3^{-1} = T_3$

Under the transformation T_3; T_1 and T_2 get diagonalised and K gets block diagonalised; this result is a direct consequence of a theorem in matrix theory (38).

Thus the stiffness eqs. (1) reduces to -

$$T_3KT_3T_3X = T_3\,F$$

or $K'X' = F'$

where
$$K' = T_3KT_3$$
$$X' = T_3X = \frac{1}{2}\begin{bmatrix} X_1+X_2+X_3+X_4 \\ X_1-X_2+X_3-X_4 \\ X_1+X_2-X_3-X_4 \\ X_1-X_2-X_3+X_4 \end{bmatrix}$$
$$F' = T_3F$$

$$F' = \frac{1}{2} \begin{bmatrix} F_1 + F_2 + F_3 + F_4 \\ F_2 - F_2 + F_3 - F_4 \\ F_1 + F_2 - F_3 - F_4 \\ F_1 - F_2 - F_3 + F_4 \end{bmatrix}$$

writting in detail -

$$K' = \frac{1}{4} \begin{bmatrix} I & I & I & I \\ I & -I & I & -I \\ I & I & -I & -I \\ I & -I & -I & I \end{bmatrix} \begin{bmatrix} A & B & C & D \\ B & A & D & C \\ C & D & A & B \\ D & C & B & A \end{bmatrix} \begin{bmatrix} I & I & I & I \\ I & -I & I & -I \\ I & I & -I & -I \\ I & -I & -I & I \end{bmatrix}$$

$$= \begin{bmatrix} A+B+C+D & & & \\ & A-B+C-D & & 0 \\ & & A+B-C-D & \\ & 0 & & A-B-C+D \end{bmatrix}$$

Therefore $K'X' = F'$ reduces to following four problems:

$$(A+B+C+D)\ X_1' = F_1' \qquad (10)$$

$$(A-B+C-D)\ X_2' = F_2' \qquad (11)$$

$$(A+B-C-D)\ X_3' = F_3' \qquad (12)$$

$$(A-B-C+D)\ X_4' = F_4' \qquad (13)$$

That the 4n x 4n problem reduces to 4 problems of n x n, which is a much simpler job than the total problem. The simplification is astronomical if the stiffness matrix is well populated. Now the following theorems can be stated.

THEOREM-1:

If a structural system has got two orthogonal planes of symmetry such that these planes divide the system into 4 regions R_1, R_2, R_3 and R_4 which are mirror images of each other w.r.t. the two planes, then one can find a set of 4-global co-ordinate systems corresponding to each region in which the stiffness matrix assumes the following form:

$$K = \begin{bmatrix} A & B & C & D \\ B & A & D & C \\ C & D & A & B \\ D & C & B & A \end{bmatrix}$$

where A, B, C, and D are n x n matrices.

THEOREM-2:

The stiffness equation for a structural system with two orthogonal planes of symmetry breaks into 4 - stiffness equations. each of order 1/4th of the

actual problem.

i.e. $[K]_{4n\times 4n} \; [X]_{4n\times 1} = [F]_{4n\times 1}$ breaks to

$$[(A+B+C+D)]_{n\times n} \; [X'_1]_{n\times 1} = [F'_1]_{n\times 1}$$

$$[(A-B+C-D)]_{n\times n} \; [X'_2]_{n\times 1} = [F'_2]_{n\times 1}$$

$$[(A+B-C-D)]_{n\times n} \; [X'_3]_{n\times 1} = [F'_3]_{n\times 1}$$

$$[(A-B-C+D)]_{n\times n} \; [X'_4]_{n\times 1} = [F'_4]_{n\times 1}$$

After solving for X'_1, X'_2, X'_3 and X'_4 one can easily find X_1, X_2, X_3 and X_4 by simple additions because,

$$X' = T_3 X \quad \text{or} \quad T_3 X' = T_3^2 X$$

But $T_3^2 = I$

So, $X = T_3 X'$

or

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} X'_1 + X'_2 + X'_2 + X'_4 \\ X'_1 - X'_2 + X'_3 - X'_4 \\ X'_1 + X'_2 - X'_3 - X'_4 \\ X'_1 - X'_2 - X'_3 + X'_4 \end{bmatrix}$$

Corollary 1

Natural frequencies of a structural system of 4n degrees of freedom with two orthogonal reflection planes can be found by solving 4 characteristic determinants of order n x n instead of solving one 4n x 4n determinant.

Proof:

Using theorem 2, the eqs. of motion for free vibrations $KX = -M\ddot{X}$, reduces to

$$(A+B+C+D)\, X_1' = -(m_1 + m_2 + m_3 + m_4)\, \ddot{X}_1'$$

$$(A-B+C-D)\, X_2' = -(m_1 - m_2 + m_3 - m_4)\, \ddot{X}_2'$$

$$(A+B-C-D)\, X_3' = -(m_1 + m_2 - m_3 - m_4)\, \ddot{X}_3'$$

$$(A-B-C+D)\, X_4' = -(m_1 - m_2 - m_3 + m_4)\, \ddot{X}_4'$$

where

$$M = \begin{bmatrix} m_1 & m_2 & m_3 & m_4 \\ m_2 & m_1 & m_4 & m_3 \\ m_3 & m_4 & m_1 & m_2 \\ m_4 & m_3 & m_2 & m_1 \end{bmatrix}$$

From Archer's(39) theory of consistent mass matrix and from the theorem -1 of this section.

$$K = \begin{bmatrix} A & B & C & D \\ B & A & D & C \\ C & D & A & B \\ D & C & B & A \end{bmatrix}$$

Let $X'_k = X'^{o}_k e^{iwt}$

Therefore $\ddot{X}'_k = - w^2 X'^{o}_k$

The above four equations reduce to -

$$\left[A+B+C+D - w^2 (m_1 + m_2 + m_3 + m_4)\right] \left[X'^{o}_1\right] = 0$$

$$\left[A-B+C-D - w^2 (m_1 - m_2 + m_3 - m_4)\right] \left[X'^{o}_2\right] = 0$$

$$\left[A+B-C-D - w^2 (m_1 + m_2 - m_3 - m_4)\right] \left[X'^{o}_3\right] = 0$$

$$\left[A-B-C+D - w^2 (m_1 - m_2 - m_3 + m_4)\right] \left[X'^{o}_4\right] = 0$$

For non-trivial solutions of these eqs. one must have

$$\det \left| A + B + C + D - w^2 (m_1 = m_2 + m_3 + m_4)\right| = 0 \quad \cdots (14)$$

$$\det \left| A - B + C - D - w^2 (m_1 - m_2 + m_3 - m_4)\right| = 0 \quad \cdots (15)$$

$$\det \left| A + B - C - D - w^2 (m_1 + m_2 - m_3 - m_4)\right| = 0 \quad \cdots (16)$$

$$\det \left| A - B - C + D - w^2 (m_1 - m_2 - m_3 + m_4)\right| = 0 \quad \cdots (17)$$

This is exactly what was needed.

Corollary - 2:

If a structural system with 4n degrees of freedom along with the axial loads on the members have got two planes of symmetry, then the buckling loads

of the structural system can be obtained by solving the four determinants of order n x n given by:

$$\det \left| A + B + C + D \right| = 0$$

$$\det \left| A - B + C - D \right| = 0$$

$$\det \left| A + B - C - D \right| = 0$$

$$\det \left| A - B - C + D \right| = 0$$

where A, B, C, D are submatrices as that of theorem-1 and are also functions of axial loads.

Proof:

Proof is exactly similar to that of corollary-1. In order to illustrate these ideas,a few examples will be solved.

EXAMPLE - 1

Vibration of orthogonal cable system by lumping the masses at nodes, shown in Fig. (3.7).

Let T = the tension in the cables. Here the only degrees of freedoms are those perpendicular to the plane of cables, and hence there is no problem with regard to choice of co-ordinate systems.

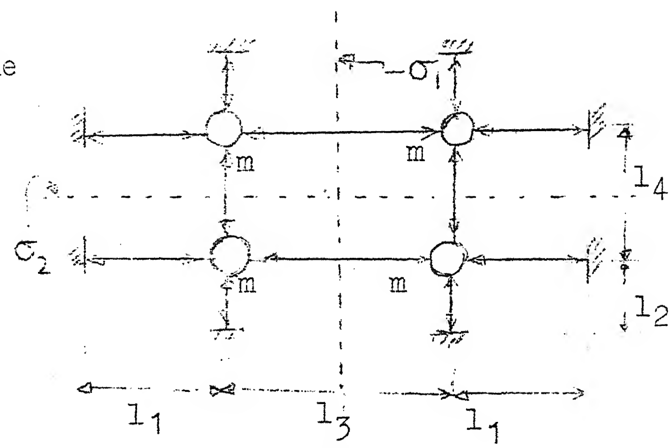


Fig. 3.7

The stiffness matrix is (there are 4-degrees of freedom):

$$K = T \begin{bmatrix} \sum \frac{1}{l_i} & -\frac{1}{l_4} & -\frac{1}{l_3} & 0 \\ -\frac{1}{l_4} & \sum \frac{1}{l_i} & 0 & -\frac{1}{l_3} \\ -\frac{1}{l_3} & 0 & \sum \frac{1}{l_i} & -\frac{1}{l_4} \\ 0 & -\frac{1}{l_3} & -\frac{1}{l_4} & \sum \frac{1}{l_i} \end{bmatrix}, \qquad M = \begin{bmatrix} m & & & 0 \\ & m & & \\ & & m & \\ 0 & & & m \end{bmatrix}$$

Equation of motion $KX + M\ddot{X} = F(t)$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, $F(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \\ f_4(t) \end{bmatrix}$

Here

$$A = T \sum \frac{1}{l_i}$$

$$B = -\frac{T}{l_4}$$

$$C = -\frac{T}{l_3}$$

$$D = 0$$

From corollary 1 one gets for free vibration.

$(A+B+C+D)\, x_1'^{\circ} = mw^2 x_1'^{\circ}$ or $w^2 = \frac{T}{m}\left(\frac{1}{l_1} + \frac{1}{l_2}\right)$

$(A-B+C-D)\, x_2'^{\circ} = mw^2\, x_2'^{\circ}$ or $w^2 = \frac{T}{m}\left(\frac{1}{l_1} + \frac{1}{l_2} + \frac{2}{l_4}\right)$

$(A-B-C+D)\, x_3'^{\circ} = mw^2\, x_3'^{\circ}$ or $w^2 = \frac{T}{m}\left(\frac{1}{l_1} + \frac{1}{l_2} + \frac{2}{l_3} + \frac{2}{l_4}\right)$

$(A+B-C-D)\, x_4'^{\circ} = mw^2\, x_4'^{\circ}$ or $w^2 = \frac{T}{m}\left(\frac{1}{l_1} + \frac{1}{l_2} + \frac{2}{l_3}\right)$

Thus all natural frequencies are obtained without solving any determinantal equation.

Mode shapes are respectively,

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

EXAMPLE - 2 VIBRATION OF 4 x 4 CABLE NETWORK

As shown in Fig. (3.8) the cable network has 16 nodes and two planes of symmetry. The transverse vibration is considered. Here the choice of co-ordinate systems for regions R_1, R_2, R_3 and R_4 is trivial because the reflection planes are parallel to the direction of motion.

Recently B.P. Singh has analysed cable network (41) for establishing cable membrane analogy. He could analyse a cable network only of 10 x 10 because of computer's limitations. The present method can easily handle a 20 x 20 network with same effort as that for 10 x 10 of B.P. Singh. Also noteworthy is the fact that even more general system than that of B.P. Singh can be handled by hand computations. e.g. a 2 x 2 network

needs to solve no determinantal equation, a 4 x 4 network needs to solve only four 4 x 4 determinants to get all frequencies, which would otherwise have needed an enormous efforts of computer even. As pointed-out earlier, the consistent mass matrix of Archer (39) will have the same symmetry as stiffness matrix, the present method is applicable. For the sake of simplicity, however, here the masses are lumped and hence the mass matrix has the form,

$$M = \begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & m \end{bmatrix} \quad \text{where} \quad m = \begin{bmatrix} m_1 & & & 0 \\ & m_2 & & \\ & & m_3 & \\ 0 & & & m_4 \end{bmatrix}$$

$$K = \begin{bmatrix} A & B & C & D \\ B & A & D & C \\ C & D & A & B \\ D & C & B & A \end{bmatrix}$$

A, B, C are 4 x 4 matrices and are given in table (3.2), D = a null matrix.

According to corollary 1 the following four characteristic equations are obtained.

TABLE 3.2

VARIOUS MATRICES RELATED TO CABLE NETWORK OF FIGURE 3.8

(multiplied by T: the tension in the cables)

Matrices	
A+B+C	$\begin{matrix} \frac{1}{l_1}+\frac{1}{l_2}+\frac{1}{l_4}+\frac{1}{l_5} & -\frac{1}{l_2} & -\frac{1}{l_5} & 0 \\ -\frac{1}{l_2} & \frac{1}{l_2}+\frac{1}{l_4}+\frac{1}{l_5} & 0 & -\frac{1}{l_5} \\ -\frac{1}{l_5} & 0 & \frac{1}{l_1}+\frac{1}{l_2}+\frac{1}{l_5} & -\frac{1}{l_2} \\ 0 & -\frac{1}{l_5} & -\frac{1}{l_2} & \frac{1}{l_2}+\frac{1}{l_5} \end{matrix}$
A+B-C	$\begin{matrix} \frac{1}{l_1}+\frac{1}{l_2}+\frac{1}{l_4}+\frac{1}{l_5} & -\frac{1}{l_2} & -\frac{1}{l_5} & 0 \\ -\frac{1}{l_2} & \frac{1}{l_2}+\frac{2}{l_3}+\frac{1}{l_4}+\frac{1}{l_5} & 0 & -\frac{1}{l_5} \\ -\frac{1}{l_5} & 0 & \frac{1}{l_1}+\frac{1}{l_2}+\frac{1}{l_5} & -\frac{1}{l_2} \\ 0 & -\frac{1}{l_5} & -(1/l_2) & \frac{1}{l_2}+\frac{2}{l_3}+\frac{1}{l_5} \end{matrix}$
A-B+C	$\begin{matrix} \frac{1}{l_1}+\frac{1}{l_2}+\frac{1}{l_4}+\frac{1}{l_5} & -\frac{1}{l_2} & -\frac{1}{l_5} & 0 \\ -\frac{1}{l_2} & \frac{1}{l_2}+\frac{1}{l_4}+\frac{1}{l_5} & 0 & -\frac{1}{l_5} \\ -\frac{1}{l_5} & 0 & \frac{1}{l_1}+\frac{1}{l_2}+\frac{1}{l_5}+\frac{2}{l_6} & \\ 0 & -\frac{1}{l_5} & -\frac{1}{l_2} & \frac{1}{l_2}+\frac{1}{l_5}+\frac{1}{l_6} \end{matrix}$

Contd. page

Continued Table 3.2

Matrices

	$\frac{1}{l_1}+\frac{1}{l_2}+\frac{1}{l_4}+\frac{1}{l_5}$	$-(1/l_2)$	$-(1/l_5)$	0
	$-(1/l_2)$	$\frac{1}{l_2}+\frac{2}{l_3}+\frac{1}{l_4}+\frac{1}{l_5}$	0	$-(1/l_5)$
A-B-C	$-(1/l_5)$	0	$\frac{1}{l_1}+\frac{1}{l_2}+\frac{1}{l_5}+\frac{1}{l_6}$	$-(1/l_2)$
	0	$-(1/l_5)$	$-(1/l_2)$	$\frac{1}{l_3}+\frac{2}{l_3}+\frac{1}{l_5}+\frac{?}{l_6}$

$$\left| A + B + C - w^2 m \right| = 0$$

$$\left| A - B + C - w^2 m \right| = 0$$

$$\left| A - B - C - w^2 m \right| = 0$$

$$\left| A - B - C - w^2 m \right| = 0$$

The matrices $A \pm B \pm C - w^2 m$ are given in Table (3.2).

The characteristic equation is

$$u^4 - u^3 \left(\frac{a}{m_1} + \frac{b}{m_2} + \frac{c}{m_3} + \frac{d}{m_4} \right) +$$

$$+ u^2 \left(\frac{ab - a_2^2}{m_1 m_2} + \frac{cd - a_2^2}{m_3 m_4} + \frac{ac - a_5^2}{m_1 m_3} + \frac{bd - a_5^2}{m_2 m_4} + \frac{bc}{m_2 m_3} + \frac{bd}{m_2 m_4} \right)$$

$$- u \left(\frac{abc - a_2^2 c - a_5^2 b}{m_1 m_2 m_3} + \frac{acd - a_2^2 a - a_5^2 d}{m_1 m_2 m_4} + \frac{abd - a_2^2 d - a_5^2 a}{m_1 m_2 m_4} + \frac{bcd - a_2^2 b - a_5^2 c}{m_2 m_3 m_4} \right)$$

$$+ \frac{a_5^4 + abcd - 2a_2^2\, a_5^2 - a_5^2(ac+db) - a_2^2(ab+cd)}{m_1\, m_2\, m_3\, m_4} = 0$$

where $u = w^2$; $a = \frac{1}{l_1} + \frac{1}{l_2} + \frac{1}{l_4} + \frac{1}{l_5}$ $\qquad a_2 = \frac{1}{l_2}$

$$b = \frac{1}{l_2} + \frac{1}{l_3} + \frac{1}{l_4} + \frac{1}{l_5} \pm \frac{1}{l_3} \qquad a_5 = \frac{1}{l_5}$$

$$c = \frac{1}{l_1} + \frac{1}{l_2} + \frac{1}{l_5} + \frac{1}{l_6} \pm \frac{1}{l_6}$$

$$d = \frac{1}{l_2} + \frac{1}{l_3} + \frac{1}{l_5} + \frac{1}{l_6} \pm \frac{1}{l_3} \pm \frac{1}{l_6}$$

All the 16-frequencies can be found by solving the above bi-quadratic equation by taking four sets of values of a, b, c and d.

OBSERVATIONS:

If X_1, X_2, X_3, X_4 and F_1, F_2, F_3, F_4 are the displacement and force vectors for the regions R_1, R_2, R_3 and R_4 respectively, then the equations of equilibrium of the cable system $KX = F$ breaks into following four equations:

$$(A+B+C)\ (X_1 + X_2+X_3+X_4) = F_1+F_2+F_3+F_4 \qquad \text{(a)}$$

$$(A-B+C)\ (X_1 - X_2+X_3-X_4) = F_1-F_2+F_3-F_4 \qquad \text{(b)}$$

$$(A-B-C)\ (X_1 - X_2-X_3+X_4) = F_1-F_2-F_3+F_4 \qquad \text{(c)}$$

$$(A+B-C)\ (X_1+X_2 -X_3-X_4) = F_1+F_2-F_3-F_4 \qquad \text{(d)}$$

The above equations physically correspond to the subsystems shown in Figs. 3.8(a), (b), (c) and (d) respectively. It can be seen from the Fig. (3.8a) that the first of the above equations will give the lowest frequency of the system because it is the subsystem which is least restrained. Hence if one is interested only in the lowest frequency one can solve for it by power method simply by taking the 1st equation of the above equations. The amount of the effort saved is enormous.

In order to determine the deflections at all 16 nodes under the applications of loads at these nodes, one needs to solve now 4 sets of 4-equations. The bandedness of the matrix K is not all affected and hence much efforts can be saved even by using Gauss elimination method.

Interesting but obvious cases are the following:

(i) $F_1=F_2=F_3=F_4$	$X_1 = X_2 = X_3 = X_4$	need to solve only eq. (a)
(ii) $F_1=F_2$, $F_2=F_4$	$X_1=X_2$, $X_3 = X_4$	need to solve eqs. (a) and (d)
(iii) $F_1=F_3$, $F_2=F_4$	$X_1=X_3$; $X_2=X_4$	need to solve eqs. (a) and (b)
(iv) $F_1=F_4$, $F_2=F_3$	$X_1=X_4$; $X_2=X_3$	need to solve eqs. (a) and (c)
(v) $F_1+F_2 = F_3+F_4$	$X_1+X_2 = X_3+X_4$	need to solve eqs. (a),(b) and (c)
(vi) $F_1+F_3 = F_2+F_4$	$X_1+X_3 = X_2+X_4$	need to solve eqs. (a), (c) and (d)
(vii) $F_1+F_4 = F_2+F_3$	$X_1+X_4 = X_2+X_3$	need to solve eqs. (a), (b) and (d)

As can be seen from the table (3.2), if $l_1 = l_2 = l_3 = l_4 = l_5 = l_6$, one gets additional symmetry in matrices, i.e. the elements of the matrices

A+B+C+D are generated by only two different elements. What does thia additional symmetry do is not clear from the present approach. This is a subject matter of Vth Chapter where group theory will tell the results of additional symmetries from the out set. However it is very clear that under such situation the frequencies given by eqs. (b) and (d) are similar i.e. the 4-roots are are doubly degenerate.

The above arguments are valid for any cable network with double-orthogonal planes of symmetry provided nodes do not come on the planes of symmetry.

EXAMPLE - 3 PLANER ORTHOGONAL GRID SYSTEM

Consider the grid system shown in Fig. (3.9). This has symmetry elements (E, $\sigma_1, \sigma_2, \sigma_3, \sigma_4, C_4, C_4, C_4^3$) but only σ_1 and σ_2 will be used.

In this example there are only rotations and moments and torsion is neglected.

The sub-matrices of stiffness matrix are given in Table (3.3)

$$A = \frac{EI}{l}\left[\begin{array}{cc|cc} 4 & 2 & \multicolumn{2}{c}{0} \\ 2 & 8 & & \\ \hline \multicolumn{2}{c|}{0} & 4 & 2 \\ & & 2 & 8 \end{array}\right], \quad B = \frac{EI}{l}\left[\begin{array}{c|cc} 0 & \multicolumn{2}{c}{0} \\ \hline 0 & 0 & 0 \\ & 0 & 2 \end{array}\right],$$

TABLE 3.3

THE TOTAL STIFFNESS MATRIX FOR THE GRID SHOWN IN FIG. 3.9

(Considering double reflection symmetry)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	4	2														
2	2	8								2						
3			4	2												
4			2	8				2								
5					4	2										
6					2	8								2		
7							4	2								
8				2			2	8								
9									4	2						
10		2							2	8						
11											4	2				
12											2	8				2
13													4	2		
14						2							2	8		
15															4	2
16												2			2	8

$$C = \frac{EI}{l}\left[\begin{array}{cc|cc} 0 & 0 & & \\ 0 & 2 & \multicolumn{2}{c}{0} \\ \hline \multicolumn{2}{c|}{0} & \multicolumn{2}{c}{0} \\ & & & \end{array}\right]$$

$$D = 0$$

$$A+B+C = \frac{2EI}{l^3}\left[\begin{array}{cc|cc} 2 & 1 & & \\ 1 & 5 & \multicolumn{2}{c}{0} \\ \hline & & 2 & 1 \\ \multicolumn{2}{c|}{0} & 1 & 5 \end{array}\right]$$

$$A - B + C = 2\,\frac{EI}{l^3}\left[\begin{array}{cc|cc} 2 & 1 & & \\ 1 & 3 & \multicolumn{2}{c}{0} \\ \hline & & 2 & 1 \\ \multicolumn{2}{c|}{0} & 1 & 5 \end{array}\right]$$

$$A+B-C = \frac{2EI}{l}\left[\begin{array}{cc|cc} 2 & 1 & & \\ 1 & 4 & \multicolumn{2}{c}{0} \\ \hline & & 2 & 1 \\ \multicolumn{2}{c|}{0} & 1 & 3 \end{array}\right]$$

$$A-B-C = \frac{2EI}{l}\left[\begin{array}{cc|cc} 2 & 1 & & \\ 1 & 3 & \multicolumn{2}{c}{0} \\ \hline & & 2 & 1 \\ \multicolumn{2}{c|}{0} & 1 & 3 \end{array}\right]$$

Their inverses are

$$(A+B+C)^{-1} \quad \frac{1}{2EI}\left[\begin{array}{cc|cc} (5/9) & -(1/9) & & \\ -(1/9) & +(2/9) & \multicolumn{2}{c}{0} \\ \hline & & (5/9) & -(1/9) \\ \multicolumn{2}{c|}{0} & -(1/9) & (2/9) \end{array}\right]$$

$$(A-B+C)^{-1} = \frac{1}{2EI}\begin{bmatrix} (3/8) & -(1/5) & 0 & 0 \\ -(1/5) & (2/5) & 0 & 0 \\ 0 & 0 & (5/9) & -(1/9) \\ 0 & 0 & -(1/9) & (2/9) \end{bmatrix}$$

$$(A+B-C)^{-1} = \frac{1}{2EI}\begin{bmatrix} (5/9) & -(1/9) & 0 & 0 \\ -(1/9) & (2/9) & 0 & 0 \\ 0 & 0 & (3/5) & -(1/5) \\ 0 & 0 & -(1/5) & (2/5) \end{bmatrix}$$

$$(A-B-C)^{-1} = \frac{1}{2EI}\begin{bmatrix} (3/5) & -(1/5) & 0 & 0 \\ -(1/5) & (2/5) & 0 & 0 \\ 0 & 0 & (3/5 & -(1/5) \\ 0 & 0 & -(1/5) & (2/5) \end{bmatrix}$$

Once the inverse has been found the bending problem $KX=F$ is atonce solved.

As an example let

$$F_1 = \begin{bmatrix} 10 \\ 0 \\ 0 \\ 10 \end{bmatrix}, \quad F_2 = F_3 = F_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Then $X_1 + X_2 + X_3 + X_4 = \frac{1}{2EI}\begin{bmatrix} (50/9) \\ -(10/9) \\ -(10/9) \\ (20/9) \end{bmatrix}$

$$X_1 - X_2 + X_3 - X_4 = \frac{1}{2EI}\begin{bmatrix} 6 \\ -2 \\ -(10/9) \\ (20/9) \end{bmatrix}$$

$$X_1 + X_2 - X_3 - X_4 = \frac{1}{2EI}\begin{bmatrix}(50/9)\\-(10/9)\\-2\\4\end{bmatrix}$$

$$X_1 - X_2 - X_3 + X_4 = \frac{1}{2EI}\begin{bmatrix}6\\-2\\-2\\4\end{bmatrix}$$

Therefore $X_1 \; \frac{1}{72EI}\begin{bmatrix}208\\-56\\-56\\112\end{bmatrix}$, $X_2 = \frac{1}{72EI}\begin{bmatrix}-8\\16\\0\\0\end{bmatrix}$,

$$X_3 = \frac{1}{72EI}\begin{bmatrix}0\\0\\16\\-32\end{bmatrix}, \quad X_4 = \frac{1}{72EI}\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

Once the displacements are known, one can determine the bending moments and shear force diagrams etc. The above procedure remains unaltered if applied to vibration problem because the consistent/lumped (or) mass matrix will have exactly same gross symmetries as that of stiffness matrix and corollary-1 of this section is applicable. (The procedure could be applied to big structures while solving them by computer. The basic purpose here was illustration of the procedure and not solving any big problem).

3.3 m-FOLD CYCLIC SYMMETRY

Here the structural system with a C_m, will be considered. They may and may not have σ's. Let the structure be partitioned in such a way that there are m-substructures and when a rotation by a mutiple of $\frac{2\pi}{m}$ is performed the structure goes into itself; e.g. a hexagonal frame (Fig. 2.15) when rotated by 60°, 120°, 180° - - - 360° it coincides with itself.

Let there are n-degrees of freedom in each sub-system. (Assume: no node common to two sub-structures).

Let X_1, X_2, X_3 - - - - X_m are the displacement vectors, F_1, F_2, F_3, - - - - F_m are the force vectors in sub-structures 1, 2, - - - m respectively w.r.t. their respective cyclically symmetric global co-ordinate systems.

Then the total displacement and force vectors for the structure are:

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_m \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_m \end{bmatrix}$$

where $X_i = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}_i$ and $F_i = \begin{bmatrix} f_i \\ f_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ f_n \end{bmatrix}$ (i=1,2...m)

Then the stiffness equation is

$$KX = F \qquad \ldots \qquad ,\ldots \quad (1)$$

Which is written more explicity as:

$$\begin{bmatrix} K_{11} & K_{12} & ---- & K_{1m} \\ K_{21} & K_{22} & --- & K_{2m} \\ -- & - & - & - \\ K_{m1} & K_{m2} & --- & K_{mm} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ X_m \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ F_m \end{bmatrix}$$

where K_{ij} = n x n matrices

i, j =1, 2, --- m

and $K_{ij} = K^T_{ji}$ (Green's - Maxwell reciprocal theorem).

Now if $\begin{pmatrix} X_1 & X_2 & --- & X_m \\ F_1 & F_2 & --- & F_m \end{pmatrix}$ are changed in cyclic

or anticyclic manner, the operations will correspond to a rotation by $\pm \frac{2\pi}{m}$ for every interchange.

Consider first the cyclic-interchanges. Let T_1, T_2, - - - T_m are operators such that

$$T_1 \begin{bmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ X_m \end{bmatrix} = \begin{bmatrix} X_m \\ X_1 \\ X_2 \\ \cdot \\ X_{m-1} \end{bmatrix} \qquad T_2 \begin{bmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ \cdot \\ X_m \end{bmatrix} = \begin{bmatrix} X_{m-1} \\ X_m \\ X_1 \\ \cdot \\ \cdot \\ X_{m-2} \end{bmatrix} \quad \ldots\ldots$$

$$\ldots T_r \begin{bmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ X_m \end{bmatrix} = \begin{bmatrix} X_{m-r+1} \\ X_{m-r+2} \\ \cdot \\ \cdot \\ X_m \\ X_1 \\ X_{m-r} \end{bmatrix} \ldots T_m \begin{bmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ \cdot \\ X_m \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ \cdot \\ X_m \end{bmatrix}$$

In a similar way one can have a set of operators A_1, A_2, - - - A_m such that;

$$A_1 \begin{bmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ \cdot \\ X_m \end{bmatrix} = \begin{bmatrix} X_2 \\ X_3 \\ \cdot \\ \cdot \\ \cdot \\ X_m \\ X_1 \end{bmatrix} \qquad A_2 \begin{bmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ X_m \end{bmatrix} = \begin{bmatrix} X_3 \\ X_4 \\ \cdot \\ \cdot \\ X_m \\ X_1 \\ X_2 \end{bmatrix}$$

$$\cdots A_r \cdot \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_m \end{bmatrix} = \begin{bmatrix} X_{r+1} \\ X_{r+2} \\ \vdots \\ X_m \\ X_1 \\ X_r \end{bmatrix} - - - A_m \cdot \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_m \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_m \end{bmatrix}$$

A matrix representation of T_1, T_2 - - - T_m and A_1, A_2 - - - A_m.

Can be easily seen to be,

$$T_1 = \begin{bmatrix} & & & & I \\ I & & & & \\ & I & & & \\ & & \ddots & & \\ & & & I & 0 \end{bmatrix} \qquad T_2 = \begin{bmatrix} & & & I & \\ & & & & I \\ I & & & & \\ & I & & \ddots & \\ \hdashline & & & I & \end{bmatrix}$$

$$T_3 \begin{bmatrix} & & & I & & \\ & & & & I & \\ & & & & & I \\ I & & & & & \\ \hdashline & & & I & & \end{bmatrix}$$

$$T_r = \left[\begin{array}{cccccccc} & & & & I & & & \\ & & & & & I & & \\ & & & & & & \cdots & \\ & & & & & & & I \\ I & & & & & & & \\ & I & & & & & & \\ & & \cdots & & & & & \\ & & & & I & & & \end{array}\right] \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} r \\ \left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} m-r \end{array}$$

$$T_m = \left[\begin{array}{ccccc} I & 0 & \cdots & & 0 \\ & I & & & \\ & & I & & 0 \\ & & & \ddots & \\ 0 & & & & I \end{array}\right]$$

$$\dots (2)$$

and

$$A_1 = \left[\begin{array}{c|cccc} & I & & & \\ & & I & & \\ & & & \ddots & \\ & & & & I \\ \hline I & & & & \end{array}\right] \qquad A_2 = \left[\begin{array}{cc|cccc} & & I & & & \\ & & & I & & \\ & & & & \ddots & \\ & & & & & I \\ \hline I & & & & & \\ & I & & & & \end{array}\right]$$

$$A_3 = \left[\begin{array}{ccc|cccc} & & & I & & & \\ & & & & I & & \\ & & & & & \ddots & \\ & & & & & & I \\ \hline I & & & & & & \\ & I & & & & & \\ & & I & & & & \end{array}\right] \qquad A_r = \left[\begin{array}{cccccc} & & & I & & \\ & & & & I & \\ & & & & & \ddots \\ & & & & & I \\ \hline I & & & & & \\ & I & & & & \end{array}\right]$$

$$A_m = \begin{bmatrix} I & & & & \\ & I & & & \\ & & \ddots & & \\ & & & & I \end{bmatrix} \qquad \ldots (3)$$

Where I is a unit matrix of n x n.

Thus one gets two sets of mn x mn matrices $\{T_i\}$ and $\{A_i\}$ under which the system changes into itself. Properties of the sets $\{T_i\}$ and $\{A_i\}$ are following:

$$T_2 = T_1^2 \qquad . \; T_i \, T_j = T_k \text{ for any } i, j$$

$$T_3 = T_2 T_1 = T_1^3 \qquad T_i(T_j T_k) = (T_i T_j) T_k$$

$$T_4 = T_3 \, T_1 = T_1^4 \qquad T_i \, T_k = I \text{ for any } i \text{ and}$$

$$T_m = T_1^m = I \qquad \text{same } k. \qquad \ldots (4)$$

Exactly similar properties are hold by the set $\{A_i\}$.

Anticipating the matter of next chapter, the sets $\{T_i\}$ and $\{A_i\}$ form groups.

Note worthy properties of the two sets are that,

$$T_1 A_1 = I \quad \text{and} \quad T_1^T = A_1$$

$$T_2 A_2 = I \qquad T_2^T = A_2 \qquad I = \text{unit matrix of nm x nm}$$

$$T_r A_r = I \qquad T_m^T = A_m$$

$\det |T_1 A_1| = \det |T_1| \quad \det |A_1| = \det |T_1| \quad \det |T_1^T|$

$\det^2 |T_1| = 1$

$\det |T_1| = \pm 1 = 1$ (-ve sign is not allowed)

$\det |T_r| = \det |T_1^r| = 1$

$\det |T_r| = \det |A_r| = 1$

Therefore, T_r and A_r are non-singular matrices.

$$\text{and} \quad T_r^{-1} = A_r = T_r^T \qquad \dots \qquad \dots \qquad (5)$$

Now if equation (1) is operated upon by T_i or A_i one gets

$$\left.\begin{aligned} T_i K T_i^{-1} \; T_i X &= T_i F \\ A_i K A_i^{-1} \; A_i X &= A_i F \end{aligned}\right\} \quad (i = 1, 2, --- m)$$

But operations by T_i or A_i does nothing more than inter-changing substructures. That is physics of the problem does not change under T_i or A_i viz.

$$\left.\begin{aligned} \bar{K}^i \bar{X}^i &= \bar{F}^i \\ K'^i X'^i &= F'^i \end{aligned}\right\} \quad \text{and} \quad KX = F \text{ are one and the same}$$

set of equations except for the change of order of the variables.

Therefore $\bar{K}^i = K = K'^i$

$$\text{or} \quad \begin{aligned} T_i K T_i^{-1} &= K \\ A_i K A_i^{-1} &= K \end{aligned} \qquad i = 1, 2, --- m \qquad (6)$$

These are 2m - sets of n-equations out of which only one set is independent, i.e. if one set is known all others follows immediately from that one. The simple reason behind this is that,

$$\begin{aligned} T_i &= T_1^i \\ A_i &= A_1^i \\ T_i &= A_i^T = T_i^{-1} \end{aligned} \qquad \text{i.e. the group property of the sets } \{T_i\} \text{ and } \{A_i\} \qquad \ldots (7)$$

e.g. From $T_1KT_1^{-1} = K$ follows

$T_1^2 \, KT_1^{-1} \quad T_1^{-1} = T_1KT_1 = K$ i.e. $T_2KT_2^{-1} = T_1KT_1^{-1} = K$

and the $T_1^3 \, K(T_1^3)^{-1} = T_2KT_2^{-1} = K$ and so on.

Similarly $T_1KT_1^T = K$ implies

$$\begin{aligned} T_1^T \; K^T \; T_1 &= K^T \\ A_1 \; K^T \; A_1^T &= K^T \\ A_1 \; KA_1^{-1} &= K \end{aligned}$$

Where $T_1^T = A_1 = T_1^{-1}$, $A_1^T = A_1^{-1}$ and $K^T = K$ have been used.

or $A_iKA_i^{-1} = K$ also follows from $T_1KT_1^{-1} = K$.

(i=1,2,- - - m)

or
$$\begin{bmatrix} K_{mm} & K_{1m}^T & K_{2m}^T & ------ & K^T_{m-1,m} \\ K_{1m} & K_{11} & K_{12} & ---- & K_{1m-1} \\ K_{2m} & K_{12}^T & K_{22} & ------ & K_{2m-1} \\ - & - & - & - & - \\ K_{m-1,m} & K_{1m-1}^T & K_{2\,m-1}^T & ---- & K_{m-1,m-1} \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1m} \\ K_{12}^T & K_{22} & \cdots & K_{2m} \\ - & - & - & - \\ K_{1m}^T & K_{2m}^T & --- & K_{mm} \end{bmatrix}$$

Therefore,
$$K_{mm} = K_{11} = K_{22} = K_{33} ---- = K_{m-1,m-1} = B_1$$
$$K_{12} = K_{23} = K_{34} = ------- = K_{m1} = B_2$$
$$K_{13} = K_{24} = K_{35} = K_{46} = --- = K_{m2} = B_3$$
$$K_{1m} = K_{21} = K_{32} = ------- = K_{m,m-1} = B_m$$

Therefore,
$$K = \begin{bmatrix} B_1 & B_2 & B_3 & ------- & B_m \\ B_m & B_1 & B_2 & ------ & B_{m-1} \\ B_{m-1} & B_n & B_1 & ------ & B_{m-2} \\ - & - & - & - & - \\ B_2 & B_3 & B_4 & ------ & B_1 \end{bmatrix} \qquad \ldots (8)$$

With $B_m = B_2^T$, $B_{m-1} = B_3^T$, $B_{m-2} = B_4^T$ and so on .

However if instead of T_1 one uses A_1 one would get:

$$K = \begin{bmatrix} B_1 & B_2 & B_3 & - - - - & B_m \\ B_2 & B_3 & B_4 & - - - B_m & B_1 \\ B_3 & B_4 & B_5 & & B_2 \\ - - & - & - & - - - & - \\ B_m & B_1 & - - - & - - - & B_{m-1} \end{bmatrix}$$

The two forms are similar as pointed out earlier in IInd Chapter in connection with cyclically symmetric matrices. Rewritting the equation (6) in the following form one gets

$$T_i K = K T_i \qquad \text{i.e.} \quad [T_i, K] = 0$$

$$A_i K = K A_i \qquad \text{i.e.} \quad [A_i, K] = 0$$

Hence by the theorem stated (38) (which can be proved in no time), there exists a set of vectors which are simultaneously eigen-vector of T_i, A_i and K. However it has been already stated that only T_i will be taken for the present purpose. Thus one can find a vector which is simultaneously eigenvector of T_1 and K ie. $T_1 \Psi = t \Psi$, $K \Psi = k.\Psi$.

It has been seen that

$$T_1^m = I$$

$$T_1^m \Psi = t_1^m \Psi = \Psi$$

or $\qquad t_1^m = 1$

or $t_1 = \theta_k$ = the m-th root of unity = $e^{\frac{2\pi ik}{m}}$

$(k = 1, 2, - - - - m)$

One can see that

$$\left[\begin{array}{cccc|c} & & & & I \\ \hline I & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & I & \end{array}\right] \begin{bmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ X_m \end{bmatrix} = \theta_k^{m-1} \begin{bmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ X_m \end{bmatrix}$$

Note that θ_k^{m-1} = one of the eigen value of T_1.

The above equations can be satisfied if,

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \cdot \\ \cdot \\ \cdot \\ X_m \end{bmatrix} = \begin{bmatrix} I \\ I\theta_k \\ I\theta_k^2 \\ \cdot \\ \cdot \\ \cdot \\ I\theta_k^{m-1} \end{bmatrix}$$

This follows immediately from the use of $\theta_k^m = 1$.

Thus

$$\begin{bmatrix} I \\ I\theta_k \\ I\theta_k^2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ I\theta_k^{m-1} \end{bmatrix}_{nm \times n}$$

are 'n' eigen vectors of T_1 for eigen value θ_k^{m-1} $(k=1,2,\ldots m)$

Hence the matrix formed from the eigen-vectors of T_1 with eigen-values θ_k^{m-1}, each n-fold degenerate, with k=1,2,3, ... m, is,

$$\begin{bmatrix} I & I & I & --- & I \\ I\theta_1 & I\theta_2 & I\theta_3 & ---- & I\theta_m \\ I\theta_1^2 & I\theta_2^2 & I\theta_3^2 & --- & I\theta_m^2 \\ \vdots & \vdots & \vdots & & \vdots \\ I\theta_1^{m-1} & I\theta_2^{m-1} & I\theta_3^{m-1} & --- & I\theta_m^{m-1} \end{bmatrix} \quad \text{i.e. } T_{ij} = I\theta_j^{i-1}, \quad T^T_{jk} = I\theta_j^{k-1}$$

(This type of matrix was ecountered in II Chapter in connection with matrix symmetries). Before any further discussion, it is better to get acquinted with this matrix by knowing some of its properties.

Consider TT^T first.

$$(TT^T)_{ik} = \sum_{j=1}^{m} T_{ij}T^T_{jk} = T_{i1}T_{ik} + T_{i2}T^T_{2k} + T_{i3}T^T_{3k} --- + T_{im}\,T^T_{mk}$$

$$= I(\theta_1^{i-1}\ \theta_1^{k-1} + \theta_2^{i-1}\ \theta_2^{k-1}\ ---- + \theta_m^{i-1}\ \theta_m^{k-1})$$

$$= I(\theta_1^{i+k-2} + \theta_2^{i+k-2} + ----- +\theta_m^{i+k-2}$$

$$= I \sum_{j=1}^{m} \theta_j^{i+k-2}$$

Then by using the fact that if $\alpha_1, \alpha_2, --- \alpha_m$ are root of the polynomial f(x) of degree 'm' then $\sum_{k=1}^{m} \alpha_k^n$ = the coefficient of x^{-n-1} in $\frac{f'(x)}{f(x)}$ when expanded in descending powers of x (which can be proved by observing that

$$\frac{f'(x)}{f(x)} = \frac{1}{x-\alpha_1} + \frac{1}{x-\alpha_2} + ---- \frac{1}{x-\alpha_m}$$

and expanding the both sides in descending powers of x), one gets that

$$\sum_{j=1}^{m} \theta_j^n = \text{coefficient of } x^{-n-1} \text{ in } \frac{\frac{d}{dx}(x^n-1)}{x^n-1}$$

because, θ_j is the root of x^n-1.

$$\text{Therefore,} \quad \sum_{j=1}^{m} \theta_j^n = 0 \quad \text{if } n \neq 0,\ m,\ 2m,\ 3m \text{ etc.} \qquad (9a)$$

$$= m \quad \text{otherwise.}$$

$$\text{So,} \quad (TT^T)_{ik} = I \sum_{j=1}^{m} \theta_j^{i+k-2} = mI \text{ if } i+k-2=0,\ m, 2m \text{ etc.}$$

$$= 0$$

or

$$TT^T = m \left[\begin{array}{c|cccc} I & & & & \\ \hline & & & & I \\ & & & I & \\ & & \cdot^{\cdot^{\cdot}} & & \\ & I & & & \end{array}\right]$$

$$(TT^T)^2 = m^2 I \quad ------ \quad \text{where this } I \text{ is } nm \times nm.$$

or $$\left(\frac{TT^T}{m}\right)^2 = I \qquad \left(\frac{TT^T}{m}\right)^{-1} = \frac{TT^T}{m}$$

and $$\det \left| (TT^T)^2 \right| = \det \left| TT^T \right| \cdot \det \left| TT^T \right| = \det \left| m^2 I \right|$$

or $\det \left| T \right| = (\pm m)^{\frac{nm}{2}}$ where use has been made of,

$\det \left| AB \right| = \det \left| A \right| \cdot \det \left| B \right|$; $\det \left| A^T \right| = \det A$ and

$\det \left| m^2 I \right| = (m^2)^{nm}$.

This shows that T is a non-singular matrix as it should be because it is a matrix obtained from eigenvectors of T_i as columns.

At this stage one very important result can be obtained.

Consider the matrix,

$$KT = \begin{bmatrix} B_1 & B_2 & ------ & B_m \\ B_m & B_1 & ---- & B_{m-1} \\ B_{m-1} & B_m & B_1 \quad -- & B_{m-2} \\ - & - & - \; - \; - & - \; - \\ B_2 & B_3 & --- \; B_m & B_1 \end{bmatrix} \begin{bmatrix} I & I & ---- & I \\ I\theta_1 & I\theta_2 & --- & I\theta_m \\ I\theta_1^2 & I\theta_2^2 & --- & I\theta_m^2 \\ \vdots & \vdots & & \vdots \\ I\theta_1^{m-1} & I\theta_2^{m-1} & --- & I\theta_m^{m-1} \end{bmatrix}$$

$$(KT)_{ik} = \left[B_{m-i+2}, \; B_{m-i+3} \; - - - - \; B_m, B_1 - - - \; B_{m-i+1} \right] \begin{bmatrix} I \\ I\theta_k \\ I\theta_k^2 \\ \vdots \\ I\theta_k^{i-2} \\ I\theta_k^{i-1} \\ \vdots \\ I\theta_k^{m-1} \end{bmatrix}$$

$$= B_{m-i+2} + B_{m-i+3}\,\theta_k + - - - \; \theta_k + B_m\theta_k^{i-2} + B_1\theta_k^{i-1} + B_{m-1}\theta_k^{m-1}$$

$$= \theta_k^{i-1}\theta_k^{m-i+1} \; (B_{m-i+2} + B_{m-i+3}\,\theta_k + B_m\theta_k^{i-2} + B_1\theta_k^{i-1}$$

$$+ - - + B_{m-i+1}\,\theta_k^{m-1})$$

$$(\text{because } \; \theta_k^{i-1}\,\theta_k^{m-i+1} = \theta_k^m = 1)$$

$$= \theta_k^{i-1} \; (B_1\theta_k^m + B_2\,\theta_k^{m+1} + 1 \; - - - - - - . + B_m\theta_k^{m-1})$$

$$= \theta_k^{i-1} \; (B_1 + B_2\theta_k + B_3\,\theta_k^2 + - - - - + B_m\theta_k^{m-1})$$

$$= \theta_k^{i-1} \; D_k$$

$$KT = \begin{bmatrix} D_1 & D_2 & D_3 & - - - - - & D_m \\ \theta_1 D_1 & \theta_2 D_2 & \theta_3 D_3 & - - - - - & \theta_m D_m \\ \theta_1^2 D_1 & \theta_2^2 D_2 & \theta_3^3 D_3 & - - - - - & \theta_m^2 D_m \\ - - - & - - - & - - - & - - - - - & - - - \\ \theta_1^{m-1} & \theta_2^{m-1} D_2 & \theta_3^{m-1} D_3 & - - - - & \theta_m^{m-1} D_m \end{bmatrix}$$

$$= \begin{bmatrix} I & I & I & - - - - & I \\ I\theta_1 & I\theta_2 & I\theta_3 & - - - - - - & I\theta_m \\ I\theta_1^2 & I\theta_2^2 & I\theta_3^3 & - - - - & I\theta_m^2 \\ - & - & - & & - \\ - & - & - & & - \\ - & - & - & & - \\ - & - & - & & - \\ I\theta_1^{m-1} & I\theta_2^{m-1} & I\theta_3^{m-1} & - - - & I\theta_m^{m-1} \end{bmatrix} \begin{bmatrix} D_1 & & & & 0 \\ & D_2 & & & \\ & & D_3 & & \\ & & & \ddots & \\ 0 & & & & D_m \end{bmatrix} \qquad (10)$$

Therefore, $\det |KT| = \det |T| \cdot \det \begin{vmatrix} D_1 & & & 0 \\ & D_2 & & \\ & & \ddots & \\ 0 & & & D_m \end{vmatrix}$

or $\det |K| \det |T| = \det |T| \det \begin{vmatrix} D_1 & & & 0 \\ & D_2 & & \\ & & \ddots & \\ 0 & & & D_m \end{vmatrix}$

(T = non-singular)

or $\det K = \prod_{k=1}^{n} \det |D_k|$

where $D_k = B_1 + B_2\theta_k + B_3\theta_k^2 + - - - - + B_m\theta_k^{m-1}$ (11)

Thus the determinant of order nm x nm breaks in the factor of determinants of order n x n. The values of determinants are of interest in determining the Buckling loads or natural frequencies of structural systems where some determinant is

equated to zero. Buckling problems of frames are usually given by one of the two forms (42 and 43).

(a) $\bar{K}X = MX$

(b) $KX = 0$ for non trivial solutions for X i.e.

$\det\left|\bar{K} - M\right| = 0$ or $\det\left|K\right| = 0$ which applying eq. (11) get factorised in to smaller order determinants. In both of the cases the role of the symmetries of axial loads in K of $\bar{K}$ and M is important because the elements of K or K and M are dependent over axial loads also. Hence the symmetries of K or $\bar{K}$ and M are now determined by not only geometry and members aggregate symmetries but also by axial load symmetries in the members. Hence the axial loads should also be cyclically symmetric if K or $\bar{K}$ and M are to be cyclically symmetric matrices. This point is not at all cleared by any of the earlier authors (18-23) although (20, 22 and 23) derive similar formula as (11) for buckling loads from different approach. The buckling problems of the form (a) correspond to the usual free vibration problem and that the form (b) also corresponds to free vibration problems can be seen from Nowaki's (44) book. Here the symmetries of the problems are also determined by the symmetries of Mass matrix for the problems in the form (a) and by the symmetries of complicated inertia effect for the problems in the form (b).

Thus buckling loads or natural frequencies are determined by

$$\det \left| D_k \right| = 0 \qquad (k= 1,2, --- m)$$

where now D_k are $n \times n$, matrices. As an example, consider the spring mass system shown in Fig. 3.10. Total no. of degrees of freedom here = 6. Applying the present procedure, the characteristic equation determining natural frequencies breaks into three 2 x 2 determinants which can be solved in no time.

Coming back to the general problem of analysis, if one applies T^{-1} to equation (1) one gets the following:

$$T^{-1}KX = T^{-1}F \quad \text{or} \quad T^{-1}KT\,T^{-1}X = T^{-1}F \tag{12}$$

Note that T^{-1} exists because $\det \left| T \right| = 0$ has been proved. As has been shown earlier, eq. (10) that,

$$KT = T \begin{bmatrix} D_1 & & & & \\ & D_2 & & & 0 \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & D_m \end{bmatrix}$$

Hence one gets equation (12) as:

$$\begin{bmatrix} D_1 & & & & & 0 \\ & D_2 & & & & \\ & & D_3 & & & \\ & & & \ddots & & \\ 0 & & & & & D_m \end{bmatrix} \begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \\ - \\ - \\ - \\ - \\ - \\ \bar{X}_m \end{bmatrix} = \begin{bmatrix} \bar{F}_1 \\ \bar{F}_2 \\ - \\ - \\ - \\ - \\ - \\ \bar{X}_m \end{bmatrix} \quad \ldots (13)$$

where
$$\begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \\ - \\ - \\ - \\ - \\ \bar{X}_m \end{bmatrix} = T^{-1} \begin{bmatrix} X_1 \\ X_2 \\ - \\ - \\ - \\ - \\ X_m \end{bmatrix}$$

and
$$\begin{bmatrix} \bar{F}_1 \\ \bar{F}_2 \\ - \\ - \\ - \\ \bar{F}_m \end{bmatrix} = T^{-1} \begin{bmatrix} F_1 \\ F_2 \\ - \\ - \\ - \\ F_m \end{bmatrix}$$

Now the problem is to find T^{-1}. This has been found as follows:

Consider first $\begin{bmatrix} I & I \\ I\theta_1 & I\theta_2 \end{bmatrix}$ $(\theta_{1,2} = \pm 1)$

If one multiplies this by $\begin{bmatrix} I & I\theta_1 \\ I & I\theta_2 \end{bmatrix}$

One has

$$\begin{bmatrix} I & I \\ I\theta_1 & I\theta_2 \end{bmatrix} \begin{bmatrix} I & I\theta_1 \\ I & I\theta_2 \end{bmatrix} = 2 \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

Similarly one can see that,

$$\begin{bmatrix} I & I & I \\ I\theta_1 & I\theta_2 & I\theta_3 \\ I\theta_1^2 & I\theta_3^2 & I\theta_3^2 \end{bmatrix} \begin{bmatrix} I & I\theta_1^2 & I\theta_1 \\ I & I\theta_2^2 & I\theta_1 \\ I & I\theta_3^2 & I\theta_1 \end{bmatrix} = 3 \begin{bmatrix} I & & \\ & I & \\ & & I \end{bmatrix}$$

Here one has to use the eq. (9a)

i.e. $$\sum_{k=1}^{m} \theta_k^n = 0, \quad \text{if } n \neq 0,\ m,\ 2m \text{ etc.}$$

$$= m \text{ otherwise.}$$

Similarly for $T = 4n \times 4n$ one finds,

$$\begin{bmatrix} I & I & I & I \\ I\theta_1 & I\theta_2 & I\theta_3 & I\theta_4 \\ I\theta_1^2 & I\theta_2^2 & I\theta_3^2 & I\theta_4^2 \\ I\theta_1^3 & I\theta_2^3 & I\theta_3^3 & I\theta_4^2 \end{bmatrix} \cdot \begin{bmatrix} I & I\theta_1^3 & I\theta_1^2 & I\theta_1 \\ I & I\theta_2^3 & I\theta_2^2 & I\theta_2 \\ I & I\theta_3^3 & I\theta_3^2 & I\theta_3 \\ I & I\theta_4^3 & I\theta_4^2 & I\theta_4 \end{bmatrix}$$

$$= 4 \begin{bmatrix} I & & & \\ & I & & \\ & & I & \\ & & & I \end{bmatrix}$$

where one has to use again $\sum_{k=1}^{m} \theta_k^n = m$ if $m = 0, m, 2m - - -$

$= 0$ otherwise.

From looking at the patterns got above one tries following:

$$\begin{bmatrix} I & I & I & \cdots & I \\ I\theta_1 & I\theta_2 & I\theta_3 & \cdots & I\theta_m \\ I\theta_1^2 & I\theta_2^2 & I\theta_3^2 & \cdots & I\theta_m^2 \\ \vdots & \vdots & \vdots & & \vdots \\ I\theta_1^{m-1} & I\theta_2^{m-1} & I\theta_3^{m-1} & \cdots & I\theta_m^{m-1} \end{bmatrix} \begin{bmatrix} I & I\theta_1^{m-1} & I\theta_1^{m-2} & \cdots & I\theta_1 \\ I & I\theta_2^{m-1} & I\theta_2^{m-2} & \cdots & I\theta_2 \\ I & I\theta_3^{m-1} & I\theta_3^{m-2} & \cdots & I\theta_3 \\ \vdots & \vdots & \vdots & & \vdots \\ I & I\theta_m^{m-1} & I\theta_m^{m-2} & \cdots & I\theta_m \end{bmatrix}$$

$$= m \begin{bmatrix} I & & & & \\ & I & & & \\ & & I & & \\ & & & \ddots & \\ & & & & I \end{bmatrix}$$

Where one has to use the eq. 9(a) viz.

$\sum_{k=1}^{m} \theta_k^n = m$ if $k = 0, m, 2m \ldots$

$= 0$ otherwise.

$$\text{Thus } T^{-1} = (1/m) \begin{bmatrix} I & I\theta_1^{m-1} & \cdots & I\theta_1 \\ I & I\theta_2^{m-1} & \cdots & I\theta_2 \\ \vdots & \vdots & & \vdots \\ I & I\theta_m^{m-1} & \cdots & I\theta_m \end{bmatrix}$$

At this stage one can prove an algebraic identity needed later.

Let

$$T^{-1} = A = (1/m)\begin{bmatrix} I & I\theta_1^{m-1} & \cdots & I\theta_1 \\ I & I\theta_2^{m-1} & \cdots & I\theta_2 \\ \vdots & \vdots & & \vdots \\ I & I\theta_m^{m-1} & \cdots & I\theta_m \end{bmatrix}$$

Then, $TA = I$ has been seen. At the same time one also has $AT = I$. Writting this latter equation in detail one has:

$$\begin{bmatrix} I & I\theta_1^{m-1} & I\theta_1^{m-2} \cdots & I\theta_1 \\ I & I\theta_2^{m-1} & I\theta_2^{m-2} \cdots & I\theta_2 \\ \vdots & \vdots & \vdots & \vdots \\ I & I\theta_m^{m-1} & I\theta_m^{m-2} \cdots & I\theta_2 \end{bmatrix} \begin{bmatrix} I & I & \cdots & I \\ I\theta_1 & I\theta_2 & \cdots & I\theta_m \\ \vdots & \vdots & & \vdots \\ I\theta_1^{m-1} & I\theta_2^{m-1} & \cdots & I\theta_m^{m-1} \end{bmatrix}$$

$$= m \begin{bmatrix} I & & & & 0 \\ & I & & & \\ & & I & & \\ & & & \ddots & \\ 0 & & & & I \end{bmatrix}$$

$$\text{or} \quad \begin{bmatrix} I & I\theta_k^{m-1} & I\theta_k^{m-2} & \cdots & I\theta_k \end{bmatrix} \begin{bmatrix} I \\ I\theta_1 \\ I\theta_1^2 \\ \vdots \\ I\theta_1^{m-1} \end{bmatrix} = m\,\delta_{kl}$$

k-th row of A

l-th column of T.

$$\text{or} \quad 1 + \theta_k^{m-1}\theta_l + \theta_k^{m-2}\theta_l^2 + \theta_k^{m-3}\theta_l^3 + \cdots + \theta_k\theta_l^{m-1} = m\,\delta_{kl} \qquad (9b)$$

Now rewriting equation (13) one has,

$$\begin{aligned} D_1 \bar{X}_1 &= \bar{F}_1 \\ D_2 \bar{X}_2 &= \bar{F}_2 \\ &\cdots\cdots \\ &\cdots\cdots \\ D_m \bar{X}_m &= \bar{F}_m \end{aligned} \qquad \ldots (14)$$

where,

$$\begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \\ \bar{X}_3 \\ \vdots \\ \bar{X}_m \end{bmatrix} = \frac{1}{m} \begin{bmatrix} I & I\theta_1^{m-1} & \cdots & I\theta_1 \\ I & I\theta_2^{m-1} & \cdots & I\theta_2 \\ \vdots & \vdots & & \vdots \\ I & I\theta_m^{m-1} & \cdots & I\theta_m \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_m \end{bmatrix}$$

and

$$\begin{bmatrix} \bar{F}_1 \\ \bar{F}_2 \\ \bar{F}_3 \\ \cdot \\ \cdot \\ \bar{F}_m \end{bmatrix} = \frac{1}{m} \begin{bmatrix} I & I\theta_1^{m-1} & \cdots\cdots & I\theta_1 \\ I & I\theta_2^{m-1} & \cdots\cdots & I\theta_2 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ I & I\theta_m^{m-1} & \cdots\cdots & I\theta_m \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ \cdot \\ \cdot \\ \cdot \\ F_m \end{bmatrix}$$

i.e. $\bar{X}_k = (1/m)\,(X_1 + X_2\,\theta_k^{m-1} + X_3\,\theta_k^{m-2} \cdots + X_m\theta_k)$

$$\bar{F}_k = \frac{1}{m}(F_1 + F_2\theta_k^{m-1} + F_3\theta_k^{m-2} + \cdots\cdots + F_m\theta_k) \qquad (15)$$

$(k = 1, 2, 3, \ldots\ldots m)$.

Solving for $\bar{X}_k$, one applies T to this vector to get X_k

i.e. $X = T\,\bar{X}$

or $X_k = \bar{X}_1\,\theta_1^{k-1} + \bar{X}_2\theta_2^{k-1} + \bar{X}_3\theta_3^{k-1} + \cdots\cdots + \bar{X}_m\theta_m^{k-1}$, ..(16)

$(k = 1, 2, \ldots m)$

Summarising the results, the following theorems may be stated:

Theorem 1

If a structural system has m-fold cyclic symmetry, such that the system may be partitioned into 'm' identical substructures (each of n-degrees of freedom

and aving no node in common) which under rotations by $\frac{2\pi k}{m}$ (k=1, 2, . . . m) coincide with each other, then by the choice of cyclically similar global co-ordinate systems for individual sub-structures, one gets the following form for the stiffness matrix.

$$K = \begin{bmatrix} B_1 & B_2 & \cdots & B_m \\ B_m & B_1 & \cdots & B_{m-1} \\ \cdots & \cdots & \cdots & \cdots \\ B_2 & B_3 & \cdots B_m & B_1 \end{bmatrix} \quad \text{with } K = K^T$$

<u>Theorem 2:</u>

If a structural system with 'mn' degrees of freedom has cyclic symmetry "C_m", then its stiffness eq. (1) gets factorised in to m-sets of n-equations given by

$$D_k \bar{X}_k = \bar{F}_k \qquad (k = 1, 2, \ldots, m)$$

where

$$D_k = B_1 + B_2\theta_k + B_3\,\theta_k^2 + \ldots + B_m\theta_k^{m-1}$$

$$\bar{F}_k = (1/m)\,(F_1 + F_2\theta_k^{m-1} + F_3\theta_k^{m-2} + \ldots + F_m\theta_k)$$

$$X_k = (1/m)\,(X_1 + X_2\theta_k^{m-1} + X_3\theta_k^{m-2} + \ldots + X_m\theta_k)$$

or $X_k = (X_1\theta_1^{k-1} + X_2\theta_2^{k-2} + \dots \quad + X_m\theta_m^{k-1})$

Where $\theta_k = e^{\frac{2\pi ki}{m}}$ = one of the mth root of unity.

Corollary -1:

If along with structural system, the load system is also m-fold cyclically symmetric then the displacement is also cyclically symmetric and hence one needs to solve only one set of eqs. (14), viz. $D_m\bar{X}_m = \bar{F}_m$ which reduces to

$$(B_1 + B_2 + \dots \quad + B_m)\, X_1 = F_1$$

where $B_1, B_2 \dots\dots B_m$ are sub-matrices of K of theorem 1, and X_1 and F_1 are the displacement and force vectors for one of the sub-structures.

Proof:

The cyclic symmetry of load system means

$F_1 = F_2 = F_3 = F_4 = \dots\dots = F_m$

Therefore, $F_k = \frac{F_1}{m}(1 + \theta_k + \theta_k^2 + \dots\dots + \theta_k^{m-1})$

$= F_1$ if $k = m$

...From eq. (9b).

$= 0$ if $k \neq m$

Therefore the m sets of equation of (14) reduce to

$D_1 \bar{X}_1 = 0$ implying $\bar{X}_1 = 0$

$D_2 X_2 = 0$ implying $\bar{X}_2 = 0$ because D_k are non-singular matrices.

. . . .

$D_{m-1} X_{m-1} = 0$ implying $X_{m-1}=0$

$D_m \bar{X}_m = F_1$

Now from eq. (11)

$$D_m = B_1 + B_2 \theta_m + B_3\theta_m^2 + \ldots + B_m \theta_m^{m-1}$$

But $\theta_m = e^{\frac{2\pi i m}{m}} = 1$

Therefore, $D_m = B_1 + B_2 + B_3 + \ldots + B_m$

From eq. (16) one has,

$$X_k = \bar{X}_1\theta_1^{k-1} + \bar{X}_2\theta_2^{k-1} + \bar{X}_3\theta_3^{k-1} + \ldots + \bar{X}_m\theta_m^{k-1}$$

$= \bar{X}_m\theta_m^{k-1}$ because $X_1 = \bar{X}_2 = \bar{X}_3 = \ldots \bar{X}_{m-1} = 0$

as proved above and

$$X_m = (1/m)\,(X_1 + X_2\,\theta_m^{m-1} + X_3\theta_m^{m-2} + \ldots X_m\theta_m)$$

also $\theta_m = 1$.

Therefore,

$$X_k = \frac{X_1 + X_2 + X_3 + \ldots \; X_m}{m}, \quad k=1,2,\ldots m$$

i.e. $X_1 = X_2 = X_3 = X_4 = \ldots = X_m$

Therefore the displacement is also cyclically symmetric.

Hence $D_m \overline{X}_m = \overline{F}_m$ reduces to

$$(B_1 + B_2 + B_3 + \ldots \; B_m) \; \frac{(X_1 + X_2 + \ldots \; X_m)}{m} = F$$

or ($(B_1 + B_2 + \ldots + B_m) \; X_1 = F_1$ and the corollary is proved.

<u>Corollary 2</u>

The natural frequencies of a m-fold cyclically symmetric structural system with mn-degrees of freedom are determined by the roots of the determinantal equations.

$$\begin{aligned} \det \left| \overline{D}_1 \right| &= 0 \\ \det \left| \overline{D}_2 \right| &= 0 \\ \ldots \ldots \ldots \\ \det \left| \overline{D}_m \right| &= 0 \end{aligned}$$

and the corresponding mode shape are given by

$$\begin{bmatrix} X_1 \\ X_1 \theta_k \\ X_1 \theta_k^2 \\ \vdots \\ X_1 \theta_k^{m-1} \end{bmatrix} \quad (k=1, \ldots m)$$

where $D_k = B_1 - w^2 M_1 + (B_2 - w^2 M_2)\,\theta_k + \ldots (B_m - w^2 M_m)\,\theta_k^{m-1}$

where M_k = submatrices of the mass matrix M which has similar symmetries as K.

Proof:

As remarked earlier the mass matrix will have the same symmetries as that of K provided one uses Archer's(39) consistant mass matrix, (and of course for lumped masses).

i.e.

$$M = \begin{bmatrix} M_1 & M_2 & \cdot & \cdot & \cdot & \cdot & M_m \\ M_m & M_1 & \cdot & \cdot & \cdot & \cdot & M_{m-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ M_2 & M_3 & \cdot & \cdot & \cdot & \cdot & M_1 \end{bmatrix}$$

The stiffness equation for free vibration is given by

$$KX + M\ddot{X} = 0$$

or $\quad (K - w^2 M)\,X = 0$

or $\quad \bar{K}X = 0$ where $\bar{K} = K - w^2 M$

Now for non-trivial solutions one must have,

$$\det |\bar{K}| = 0 \text{ which determines w.}$$

From eq. (11) one gets

$$\det |\bar{K}| = \prod_{k=1}^{m} \det |\bar{D}_k| = 0$$

where $D_k = B_1 - w^2M_1 + (B_2-w^2 M_2)\,\theta_k + \ldots + (B_m-w^2M_m)\theta_k^{m-1}$

or $\det \left| \bar{D}_1 \right| = 0$

$\det \left| \bar{D}_2 \right| = 0$

. . . .

$\det \left| \bar{D}_m \right| = 0$ determine the w.

Note that this proof is obtained with the help of equation (11). The same could be obtained from either equation (14) or from theorem 2 of this section which are more general than equation (11).

From equation (14) it follows that $KX = 0$ means

$D_k\, X_k = 0 \qquad (k=1,2,\ldots m)$

Therefore $\bar{K}X = 0$ ($\bar{K} = K-w^2M$ has same symmetries as K)

means that

$\bar{D}_k\, \bar{X}_k = 0$

Now let $\det \left| \bar{D}_1 \right| = 0$ for some k = 1 and for all other k

$\det \quad \bar{D}_k \neq 0$

Then $\bar{X}_k = 0$ for all $k \neq 1$ because $\bar{D}_k \quad (k \neq 1)$ are non-singular matrices, viz. for frequencies obtained from $\det \left| \bar{D}_1 \right| = 0$, one has $\bar{X}_k = 0$ for $k \neq 1$.

From eq. (16) then

$$X_i = \bar{X}_1 \theta_1^{i-1} + \bar{X}_2 \theta_2^{i-1} + \dots \dots + \bar{X}_l \theta_l^{i-1} \dots + \bar{X}_m \theta_m^{i-1}$$

$$= \bar{X}_l \theta_l^{i-1}$$

Therefore $X_1 = \bar{X}_l$

$$X_2 = X_l \theta_l = X_1 \theta_l$$

$$X_3 = X_l \theta_l^2 = X_1 \theta_l^2$$

$$\dots\dots\dots\dots$$

$$X_m = X_l \theta_l^{m-1} = X_1 \theta_l^{m-1}$$

Therefore for the frequencies obtained from,

$$\det \left| \bar{D}_l \right| = 0 \quad \text{one has}$$

$$\begin{bmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ X_m \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ X_m \end{bmatrix}_l = \begin{bmatrix} X_1 \\ X_1\theta_l \\ X_1\theta_l^2 \\ \cdot \\ \cdot \\ \cdot \\ X_1\theta_l^{m-1} \end{bmatrix} = \text{lth mode-shape}$$

This holds true for any l = 1, 2, ... m.

Therefore mode shapes are given by,

$$\begin{bmatrix} X_1 \\ X_1\theta_k \\ X_1\theta_k^2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ X_1\theta_k^{m-1} \end{bmatrix}, \qquad (k=1,\ 2,\ \ldots\ldots\ m)$$

Note that for each k one has n-column vectors corresponding to the n-frequencies obtained from each det $|\bar{D}_k| = 0$

Corollary 3

The buckling loads for m-fold cyclically symmetric structural systems under m-fold cyclically symmetric axial load systems are given by

$$\det |D_k| = 0 \qquad (k=1,2,\ \ldots\ .\ m)$$

where D_k are functions of axial loads.

The corresponding buckling modes are given by,

$$\begin{bmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ \cdot \\ X_m \end{bmatrix} = \begin{bmatrix} X_1 \\ X_1\theta_k \\ X_1\theta_k^2 \\ \cdot \\ \cdot \\ X_1\theta_k^{m-1} \end{bmatrix}$$

Proof:

The proof of this corollary is exactly similar to that of corollary - 2 if one recalls the buckling loads and modes are given by

$$KX = \lambda MX \quad \text{or} \quad \bar{K}X = 0$$

Note that the so called "critical load" is given by the smallest root of det $\left| D_k \right| = 0$ for all k.

To illustrate these ideas, following examples will be considered. The example 3 of the previous section will be taken, but instead of using its double reflection symmetry, its 4-fold cyclic symmetry will be used here.

EXAMPLE 1

Consider the orthogonal grid work shown in Fig. 3.9. This system has symmetry elements ($E, \sigma_1, \sigma_2, \sigma_3, \sigma_4, C_4, C_4^2, C_4^3$). In section 3.2 only σ_1 and σ_2 were used. Here only (E, C_4, C_4^2, C_4^3) will be used and the procedure developed in this section will be applied. Then this result will be compared with that of section 3.2 and the superiority of the either method with regard to this problem will become clear.

Since this system has a 'C_4' therefore the procedure developed in this section is applicable with $m = 4$ and $n = 4$.

Therefore, stiffness matrix is

$$\begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ B_4 & B_1 & B_2 & B_3 \\ B_3 & B_4 & B_1 & B_2 \\ B_2 & B_3 & B_4 & B_1 \end{bmatrix}$$

where B_i are 4 x 4 matrices

$$B_4 = B_2^T$$

Since the sub-structure 3 has no connection with sub -tructure 1, therefore $B_3 = 0$.

$$\text{So, } K = \begin{bmatrix} B_1 & B_2 & 0 & B_2^T \\ B_2^T & B_1 & B_2 & 0 \\ 0 & B_2^T & B_1 & B_2 \\ B_2 & 0 & B_2^T & B_1 \end{bmatrix}$$

The stiffness matrix is given in Table 3.4. One has

$$B_1 = \frac{EI}{l} \left[\begin{array}{cc|cc} 4 & 2 & & \\ 2 & 8 & & \\ \hline & & 4 & 2 \\ & & 2 & 8 \end{array}\right] \qquad B_2 = \frac{EI}{l} \left[\begin{array}{cc|cc} & & & \\ \hline & & & \\ & 2 & & \end{array}\right]$$

From theorem 2 of this section the stiffness equation breaks in to:

$$D_1 \overline{X}_1 = \overline{F}_1$$

$$D_2 \overline{X}_2 = \overline{F}_2$$

TABLE 3.4

THE STIFFNESS MATRIX FOR THE GRID SHOWN IN FIG. 3.9 (Considering cyclically symmetric)

	1	2	3	4	5	5	6	7	8	9	10	11	12	13	14	15	16
1	4	2															
2	2	8															
3			4	2													
4			2	8			2										
5						4	2										
6				2		2	8										
7								4	2								
8								2	8		2						
9										4	2						
10									2	2	8						
11												4	2				
12												2	8		2		
13														4	2		
14											2			2	8		
15																4	2
16		2															

$$D_3\bar{X}_3 = \bar{F}_3 \qquad \text{where} \quad D_k = B_1 + B_2\theta_k + B_3\theta_k^2 + B_4\theta_k^3$$

$$D_4\bar{X}_4 = \bar{F}_4 \qquad\qquad = B_1 + B_2\theta_k + B_2^T\,\theta_k^3$$

$$\bar{F}_k = \frac{1}{4}(F_1 + F_2\theta_k^3 + F_3\theta_k^2 + F_4\theta_k)$$

$$\bar{X}_k = \frac{1}{4}(X_1 + X_2\theta_k^3 + X_3\theta_k^2 + X_4\theta_k)$$

$$X_k = X_1\theta_1^{k-1} + X_2\theta_2^{k-1} + X_3\theta_3^{k-1} + X_4\theta_4^{k-1}$$

where $\theta_1 = 1$, $\theta_2 = -1$, $\theta_3 = i$, $\theta_4 = -i$

Thus

$$D_1 = B_1 + B_2 + B_2^T \;; \qquad \bar{F}_1 = (1/4)\,(F_1 + F_2 + F_3 + F_4)$$

$$D_2 = B_1 - B_2 - B_2^T \qquad \bar{F}_2 = (1/4)\,(F_1 - F_2 + F_3 - F_4)$$

$$D_3 = B_1 + i\,(B_2 - B_2^T) \qquad \bar{F}_3 = (1/4)(F_1 - F_3 - i(F_2 - F_4))$$

$$D_4 = B_1 - i\,(B_2 - B_2^T) \qquad \bar{F}_4 = (1/4)\,(F_1 - F_3 + i(F_2 + F_4))$$

Therefore the stiffness equations are:

$$(B_1 + B_2 + B_2^T)\,(X_1 + X_2 + X_3 + X_4) = F_1 + F_2 + F_3 + F_4 \qquad (a)$$

$$(B_1 - B_2 - B_2^T)\,(X_1 - X_2 + X_3 - X_4) = F_1 - F_2 + F_3 - F_4 \qquad (b)$$

$$(B_1 + i(B_2 - B_2^T))\,(X_1 - X_3 - i(X_2 - X_4)) = F_1 - F_3 - i(F_2 - F_4) \qquad (c)$$

$$(B_1 - i(B_2 - B_2^T))(X_1 - X_3 + i(X_2 - X_4)) = F_1 - F_3 + i(F_2 - F_4) \qquad (d)$$

Now if one wants to work with real quantities only, the equations (c) and (d) to-gether are:

$$\begin{bmatrix} B_1 & -(B_2-B_2^T) \\ B_2-B_2^T & B_1 \end{bmatrix} \begin{bmatrix} X_1 - X_3 \\ X_2-X_4 \end{bmatrix} = \begin{bmatrix} F_1 - F_3 \\ F_2 - F_4 \end{bmatrix}$$

These equations are now 8 coupled equations while equations (a) and (b) are 4 coupled equations each. It is very clear now, that the present scheme for this example is inferior to the scheme of section 3.2 (where 4-sets of uncoupled equations were obtained) viz., if structural system has symmetry $(E, C_4, C_4^2, C_4^3, \sigma_1, \sigma_2, \sigma_3, \sigma_4)$ and if one uses only σ_1 and σ_2 one gets more simplifications in problems than that from E, C_4, C_4^2 and C_4^3 although the latter looks to be a symmetry of higher order. What would be the simplification if one uses all the 8 symmetry elements seems to be a valid question now and will be answered in latter chapters. In any case this example distinguishes the two kinds of symmetry elements which intuitively seemed to be one and the same.

Coming back to the problem,

$$B_1 + B_2 + B_2^T = \frac{EI}{l} \left[\begin{array}{cc|cc} 4 & 2 & & \\ 2 & 8 & & 2 \\ \hline & & 4 & 2 \\ & 2 & 2 & 8 \end{array}\right],$$

$$B_1 - B_2 - B_2^T = \frac{EI}{l}\left[\begin{array}{cc|cc} 4 & 2 & & \\ 2 & 8 & & -2 \\ \hline & & 4 & 2 \\ & -2 & 2 & 8 \end{array}\right]$$

$$B_2 - B_2^T = \frac{EI}{l}\left[\begin{array}{cc|cc} & & & \\ & & & -2 \\ \hline & & & \\ & 2 & & \end{array}\right]$$

One finds that these matrices are more populated than the $A \pm B \pm C$ of section 3.2(Example 3) which poses additional difficulties in their inversions. Working with complex quantities, however needs inversions of only three 4 x 4 matrices and one has

$$(B_1 + B_2 + B_2^T)^{-1} = \frac{71}{90EI}\left[\begin{array}{cc|cc} \frac{26}{7} & -1 & -(1/7) & (2/7) \\ -1 & 2 & (2/7) & -(4/7) \\ \hline -(1/7) & (2/7) & (26/7) & -1 \\ (2/7) & -(4/7) & -1 & 2 \end{array}\right]$$

$$(B_1 - B_2 - B_2^T)^{-1} = \frac{71}{90EI}\left[\begin{array}{cc|cc} (26/7) & -1 & (1/7) & -(2/7) \\ -1 & 2 & -(2/7) & (4/7) \\ \hline (1/7) & -(2/7) & (26/7) & -1 \\ -(2/7) & (4/7) & -1 & 2 \end{array}\right]$$

$$\left[B_1 + i(B_2 - B_2^T)\right]^{-1} = \frac{71}{90EI}\left[\begin{array}{cc|cc} (26/7) & -1 & (i/7) & -(2i/7) \\ -1 & 2 & -(2i/7) & (4i/7) \\ \hline -(i/7) & (2i/7) & (26/7) & -1 \\ (2i/7) & -(4i/7) & -1 & 2 \end{array}\right]$$

For a check let us take same load vector of example 3 of section 3.2, i.e.

$$F_1 = \begin{bmatrix} 10 \\ 0 \\ 0 \\ 10 \end{bmatrix}, \quad F_2 = F_3 = F_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Then, $\bar{X}_4 = X_1 + X_2 + X_3 + X_4 = \frac{1}{9EI} \cdot \begin{bmatrix} 28 \\ -11 \\ -8 \\ 16 \end{bmatrix}$

$$\bar{X}_2 = X_1 - X_2 + X_3 - X_4 = \frac{1}{9EI}\begin{bmatrix} 24 \\ -3 \\ -6 \\ 12 \end{bmatrix}$$

$$\bar{X}_3 = X_1 - X_3 - i(X_2 - X_4) = \frac{1}{9EI}\begin{bmatrix} 26 - 2i \\ -7 + 4i \\ -7 + i \\ 14 - 2i \end{bmatrix}$$

Then $X_1 - X_3 = \frac{1}{9EI} \begin{bmatrix} 26 \\ -7 \\ -7 \\ 14 \end{bmatrix}$ $X_2 - X_4 = \frac{1}{9EI} \begin{bmatrix} 2 \\ -4 \\ -1 \\ 2 \end{bmatrix}$

Therefore, $X_1 = \frac{1}{9EI} \begin{bmatrix} 26 \\ -7 \\ -7 \\ 14 \end{bmatrix}$ $X_2 = \frac{1}{9EI} \begin{bmatrix} 2 \\ -4 \\ 0 \\ 0 \end{bmatrix}$

$X_3 = \frac{1}{9EI} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $X_4 = \frac{1}{9EI} \begin{bmatrix} 0 \\ 0 \\ -1 \\ 2 \end{bmatrix}$

which exactly tallies with the result of section 3.2 but has cost more than the procedure of section 3.2.

EXAMPLE -2

Consider the free vibrations of the spring mass system shown in Fig. 3.10(a). Here the system has symmetry $(E, \sigma_1, \sigma_2, \sigma_3, C_3, C_3^2)$ but due to the limitations of the the present chapter's approach one can use only (E, C_3, C_3^2) and σ_1, σ_2, σ_3 can in no case be used.

One of the difficulties in solving such problems by hand is that of formulating the problem i.e. in determining the stiffness matrix because of complicated geometry. However if one is convinced of the theorem-1 of this section, one would find enormous ease in determining

the stiffness matrix for such system because one has to determine only about $\frac{mn^2}{2}$ elements out of m^2n^2 elements, where m and n are as used previously. The same of course holds true for any cyclically symmetric structural system. The physicists (45) determine the stiffness matrices of such systems in cut and dry fashion by putting enormous ungauranteed bull work. Usually it is easier to visualise forces than displacements and hence in such cases one should use the procedure of section 2.3 viz.

$$K = ASA^T$$

where A, S are projection and member stiffness matrices respectively.

Let us number the spring and nodal forces as shown in Fig. 3.10 (b). Then the A matrix is given by

$$\begin{bmatrix} {}_1 \\ {}_2 \\ {}_3 \\ {}_4 \\ {}_5 \\ {}_6 \end{bmatrix} = \left[\begin{array}{ccc|ccc|ccc|ccc} -1 & 0 & 1 & & & & & & & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & 0 & & & & & & & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 0 \\ \hline & & & -1 & 0 & 1 & & & & 0 & -\frac{1}{2} & \frac{1}{2} \\ & & & 0 & -1 & 0 & & & & 0 & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \\ \hline & & & & & & -1 & 0 & 1 & \frac{1}{2} & 0 & -\frac{1}{2} \\ & & & & & & 0 & -1 & 0 & \frac{\sqrt{3}}{2} & 0 & \frac{\sqrt{3}}{2} \end{array}\right] \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \\ F_9 \\ F_{10} \\ F_{11} \\ F_{12} \end{bmatrix}$$

i.e $A = \begin{bmatrix} A_1 & 0 & 0 & A_2 \\ 0 & A_1 & 0 & A_3 \\ 0 & 0 & A_1 & A_4 \end{bmatrix}$ where A_i = 2 x 3 matrices.

Also,

$$S = \begin{bmatrix} a & & & 0 \\ & a & & \\ & & a & \\ 0 & & & b \end{bmatrix} \quad \text{where } a = \begin{bmatrix} k_2 & & 0 \\ & k_3 & \\ 0 & & k_2 \end{bmatrix} \quad b = \begin{bmatrix} k_1 & & 0 \\ & k_1 & \\ 0 & & k_1 \end{bmatrix}$$

(k_1, k_2, k_3 are spring constants as shown in Fig. 3.10(a)

Thus

$$ASA^T = \begin{bmatrix} A_1 & 0 & 0 & A_2 \\ 0 & A_1 & 0 & A_3 \\ 0 & 0 & A_1 & A_4 \end{bmatrix} \cdot \begin{bmatrix} a & & & 0 \\ & a & & \\ & & a & \\ 0 & & & b \end{bmatrix} \cdot \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_1 & 0 \\ 0 & 0 & A_1 \\ A_2 & A_3 & A_4 \end{bmatrix}$$

$$= \begin{bmatrix} A_1aA_1^T + A_2bA_2^T & A_2bA_3^T & A_2bA_4^T \\ A_3bA_2^T & A_1aA_1^T + A_3bA_3^T & A_3bA_4^T \\ A_4bA_2^T & A_4bA_3^T & A_1aA_1^T + A_4bA_4^T \end{bmatrix}$$

It can be seen that $= A_k b A_j^T = k_1 A_k A_j^T$ because $b = k_1 I$

Also $\quad A_k A_k^T = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{bmatrix} \quad (k=1,\ 2,\ 3 \text{ and } 4)$

and $\quad A_1 a A_1^T = \begin{bmatrix} 2k_2 & 0 \\ 0 & k_3 \end{bmatrix}$

$$A_2 A_4^T = A_3 A_2^T = A_4 A_3^T = \begin{bmatrix} -(1/4) & -(\sqrt{3}/4) \\ (\sqrt{3}/4) & (\sqrt{3}/4) \end{bmatrix}$$

Therefore the stiffness matrix is given by

$$K = \begin{bmatrix} A & B & B^T \\ B^T & A & B \\ B & B^T & A \end{bmatrix} = \left[\begin{array}{cc|cc|cc} 2k_2+\frac{k_1}{2} & 0 & \frac{1}{4}k_1 & \frac{\sqrt{3}}{4}k_1 & \frac{1}{4}k_1 & -\frac{\sqrt{3}}{4}k_1 \\ 0 & k_3+\frac{3k_1}{2} & -\frac{\sqrt{3}}{4}k_1 & \frac{3}{4}k_1 & \frac{\sqrt{3}}{4}k_1 & \frac{3}{4}k_1 \\ \hline -\frac{1}{4}k_1 & -\frac{\sqrt{3}}{4}k_1 & 2k_2+\frac{k_1}{2} & 0 & -\frac{1}{4}k_1 & \frac{\sqrt{3}}{4}k_1 \\ \frac{\sqrt{3}}{4}k_1 & \frac{3}{4}k_1 & 0 & k_3+\frac{3k_1}{2} & -\frac{\sqrt{3}}{4}k_1 & \frac{\sqrt{3}}{4}k_1 \\ \hline -\frac{1}{4}k_1 & \frac{\sqrt{3}}{4}k_1 & -\frac{1}{4}k_1 & \frac{\sqrt{3}}{4}k_1 & 2k_2+\frac{k_1}{2} & 0 \\ -\frac{\sqrt{3}}{4}k_1 & \frac{3}{4}k_1 & \frac{\sqrt{3}}{4}k_1 & \frac{3}{4}k_1 & 0 & k_3+\frac{3k_1}{2} \end{array}\right]$$

Mass matrix is $M = \begin{bmatrix} \bar{m} & & \\ & \bar{m} & \\ 0 & & \bar{m} \end{bmatrix}$ where $\bar{m} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$

where m are masses.

Thus applying the results of corollary 2 of this section frequencies are given by:

$$\det \left| \bar{D}_k \right| = 0 \quad \text{where} \quad \bar{D}_k = \bar{A} + B\theta_k + B^T\theta_k^2 , \quad (k=1,2,3)$$

$$\bar{A} = A - mw^2$$

$$\theta_k = e^{\frac{2\pi ki}{3}}$$

$$\theta_k^2 = e^{\frac{-2\pi ki}{3}}$$

$$\bar{D}_k = \begin{bmatrix} 2k_2 + (1/2)k_1 - mw^2 & 0 \\ 0 & k_3 + \frac{3}{2}k_1 - mw^2 \end{bmatrix} + \begin{bmatrix} -\frac{1}{4}k_1 & (3/4)k_1 \\ -\frac{3}{4}k_1 & (3/4)k_1 \end{bmatrix}$$

$$\cdot\, e^{\frac{2\pi ki}{3}} + \begin{bmatrix} -(1/4)k_1 & -(3/4)k_1 \\ +(3/4)k_1 & (3/4)k_1 \end{bmatrix} \cdot e^{\frac{-2\pi ki}{3}}$$

$$= \begin{bmatrix} 2k_2 + (1/2)k_1(1-\cos\frac{2\pi k}{3}) - mw^2 & \frac{3ik_1}{2}\sin\frac{2\pi k}{3} \\ -(3/2)k_1 i \sin\frac{2\pi k}{3} & k_3 + (3/2)k_1(1+\cos\frac{2\pi k}{3}) - mw^2 \end{bmatrix}$$

where use has been made of $e^{i\theta} \pm e^{i\theta} = 2 \cdot \begin{matrix} \cos\theta \\ i\sin\theta \end{matrix}$

Now one can get all the frequencies by solving just one determinantal equation of 2 x 2 instead of big determinant.

of 6 x 6. A slight observation of $\bar{D}_k$ however shows something even more than this viz.

for k = 3; $-3k_1 \sin \frac{2\pi k}{3} = 0$ and thus the two roots are given by:

$$w_1 = \frac{2k_2}{m}, \quad w_2 = k_3 + 3k_1.$$

These roots require no solution of determinantal equation even of 2 x 2. They come purely from symmetry arguments. That these correspond to cyclically symmetric displacements will be seen very soon. For k = 1, 2, one has to solve the 2 x 2 determinantal equation,

$$\begin{vmatrix} 2k_2 + (3/4)k_1 - mw^2 & \pm (3/4)ik_1 \\ \pm (3/4)ik_1 & k_3+(3/4)k_1-mw^2 \end{vmatrix} = 0$$

or $$2mw^2 = 2k_2 + k_3 + (3/2)k_1 \pm\sqrt{4k_2^2 + k_3^2 + (9/4)k_1^2 - 4k_2k_3}$$

This is doubly degenerate root. Thus all frequencies are determined now. Mode-shapes are now given by:

for k = 3; $\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_1 \\ X_1 \end{bmatrix}$, for k=1; $\begin{bmatrix} X_1 \\ X_1 e^{\frac{2\pi i}{3}} \\ X_1 e^{-\frac{2\pi i}{3}} \end{bmatrix}$; for k=2; $\begin{bmatrix} X_1 \\ X_1 e^{-\frac{2\pi i}{3}} \\ X_1 e^{\frac{2\pi i}{3}} \end{bmatrix}$

1. Triple orthogonal planes of symmetries e.g. three dimensional orthogonal grid works.

2. Reflection-cyclic symmetries where the reflection plan is σ_h (a plane perpendicular to the axes of symmetry) e.g. the frame shown in Fig. (2.16).

3. Inversion symmetry, e.g. frames of Figs.(1.3) and (2.4).

In what follows the results will be quoted without giving proofs because the proof is trivial if one either follows the routine of sections 3.1, 3.2 and 3.3 or combines the results of appropriate sections.

Theorem 1

If a structural system has 3-orthogonal planes of symmetry not containing any node which divide the structural system in 8-regions each with n-degrees of freedom and are mirror images of each other w.r.t. one of the reflection planes, and if one chooses 8 global co-ordinate systems mirror image of each other then the stiffness matrix will have the form:

$$K = \begin{bmatrix} A & B \\ B & A \end{bmatrix} \quad \text{where } A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ A_2 & A_1 & A_4 & A_3 \\ A_3 & A_4 & A_1 & A_2 \\ A_4 & A_3 & A_2 & A_1 \end{bmatrix} \text{ and } B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ B_2 & B_1 & B_4 & B_3 \\ B_3 & B_4 & B_1 & B_2 \\ B_4 & B_3 & B_2 & B_1 \end{bmatrix}$$

or

$$K = \begin{bmatrix} C_1 & C_2 & C_3 & C_4 \\ C_2 & C_1 & C_4 & C_3 \\ C_3 & C_4 & C_1 & C_2 \\ C_4 & C_3 & C_2 & C_1 \end{bmatrix} \quad \text{where } C_i = \begin{bmatrix} A_i & B_i \\ B_i & A_i \end{bmatrix}$$

$(i=1, 2, 3, 4)$

where A_i and B_i are $n \times n$ matrices $(i=1,2,3,4)$.

As discussed in Chapter 2, these matrices may be called centro-centro symmetric matrices, of course with $A_i^T = A_i$ and $B_i^T = B_i$ due to $K = K^T$.

Theorem 2:

If a structural system satisfies the conditions of above theorem then it's stiffness equation may be factorised into 8 sets of equations viz.

$$KX = F \quad \text{where} \quad X = \begin{bmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ X_8 \end{bmatrix}, \quad F = \begin{bmatrix} F_1 \\ F_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ F_8 \end{bmatrix} \quad \text{and} \quad X_i = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}_i \quad (i=1,\ldots 8) \text{ etc.}$$

breaks into,

$$D_k \bar{X}_k = \bar{F}_k \quad (k=1,2, \ldots 8)$$

where

$$D_{1,2} = (A_1 + A_2 + A_3 + A_4) \pm (B_1 + B_2 + B_3 + B_4)$$

$$D_{3,4} = (A_1 - A_2 + A_3 - A_4) \pm (B_1 - B_2 + B_3 - B_4)$$

$$D_{5,6} = (A_1 + A_2 - A_3 - A_4) \pm (B_1 + B_2 - B_3 - B_4)$$

$$D_{7,8} = (A_1 - A_2 - A_3 + A_4) \pm (B_1 - B_2 - B_3 + B_4)$$

and corresponding $\bar{X}_k$ or $\bar{F}_k$ have similar expressions interms of $X_1, X_2 \ldots . X_8$ or $F_1, F_2, \ldots . F_8$ respectively.

All the corollaries of the types derived in section 3.1, 3.2 and 3.3, may be derived by reasonings similar to that used in the above section. If this procedure is applied to the spring mass system shown in Fig. 3.11 one can find all the 24 frequencies and mode shapes may be found in no time because then one would have to solve only 3 x 3 determinants.

Theorem 3:

If a structural system with 2·n·m degrees of freedom with a 'C_m' and a 'σ_h' such that no node lies on C_m or σ_h, (i.e. the structural system can be divided into 2m identical sub-systems each with n-degrees of freedom such that no node is common to any two sub-systems)

and if the global co-ordinate systems corresponding to each sub-system are so chosen that they coincide with each other under symmetry operations, $m\, C_m$ and σ_h then the stiffness matrix will assume the either of the following equivalent forms:

$$K = \left[\begin{array}{cccc|cccc} A_1 & A_2 & \cdots & A_m & B_1 & B_2 & \cdots\cdots & B_m \\ A_m & A_1 & \cdots\cdots & A_{m-1} & B_m & B_1 & \cdots\cdots & B_{m-1} \\ \hdashline A_2 & A_3 & \cdots\cdots & A_1 & B_2 & B_3 & \cdots\cdots & B_m \\ \hline B_1 & B_2 & \cdots\cdots & B_m & A_1 & A_2 & \cdots\cdots & A_m \\ B_m & B_1 & \cdots\cdots & B_1 & A_m & A_1 & & A_{m-1} \\ \hdashline B_2 & B_3 & \cdots\cdots & B_1 & A_2 & A_3 & \cdots\cdots & A_1 \end{array}\right]$$

or

$$K = \left[\begin{array}{cccc} C_1 & C_2 & \cdots\cdots & C_m \\ C_m & C_1 & \cdots\cdots & C_{m-1} \\ \hdashline C_2 & C_3 & \cdots\cdots & C_1 \end{array}\right] \quad \text{where } C_k = \begin{bmatrix} A_k & B_k \\ B_k & A_k \end{bmatrix}$$

Note that the above types of matrices have been called centro-cyclically and cyclically-centro symmetric matrices in the Chapter II.

Theorem - 4:

If a structural system satisfies all the conditions

of the above theorem, then its stiffness equation breaks in to:

$$D_k \bar{X}_k = \bar{F}_k$$
$$D'_k \bar{X}'_k = \bar{F}'_k \qquad (k = 1, 2, \ldots m)$$

where

$$D_k = A_1 + B_1 + (A_2 + B_2)\,\theta_k + \ldots\ldots + (A_m + B_m)\theta_k^{m-1}$$

$$D'_k = A_1 - B_1 + (A_2 - B_2)\,\theta_k + \ldots\ldots + (A_m - B_m)\theta_k^{m-1}$$

$$\bar{X}_k = X_1 + Y_1 + (X_2 + Y_2)\,\theta_k^{m-1} + \ldots\ldots + (X_m + Y_m)\,\theta_k$$

$$\bar{X}'_k = X_1 - Y_1 + (X_2 - Y_2)\,\theta_k^{m-1} + \ldots\ldots + (X_m - Y_m)\,\theta_k$$

$$\bar{F}_k = F_1 + G_1 + (F_2 + G_2)\theta_k^{m-1} + \ldots\ldots + (F_m + G_m)\,\theta_k$$

$$\bar{F}'_k = F_1 - G_1 + (F_2 - G_2)\theta_k^{m-1} + \ldots\ldots + (F_m - G_m)\,\theta_k$$

and $\theta_k = e^{\frac{2\pi ik}{m}}$

displacement vector $= \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X_1 \\ \vdots \\ X_m \\ Y_1 \\ \vdots \\ Y_m \end{bmatrix}$ and force vector $= \begin{bmatrix} F \\ G \end{bmatrix} = \begin{bmatrix} F_1 \\ \vdots \\ F_m \\ G_1 \\ \vdots \\ G_m \end{bmatrix}$

where

$$X_k = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \dot{x}_n \end{bmatrix} \quad \text{etc.} \quad (k=1,\ 2,\ \ldots.\ m)$$

Again various corollaries similar to that of sections 3.1, 3.2 and 3.3 may be derived by same kinds of arguments and problems of vibrations, buckling etc. can be simplified.

Theorem 5

If a structural system has an inversion symmetry 'i' and if inversion center does not contain any node, then for planner and space systems the theorems (1) and (2) of section 3.3 with m = 1 and theorems (3) and (4) of this section respectively, are applicable.

Note: The above theorem holds because, planner inversion is nothing but a 'C_2' and space-inversion is generated by $\sigma_h\ C_2$.

3.5 INADEQUACIES OF THE PRESENT METHODS:

It has been seen in discussions of sections 3.1, 3.2 and 3.3 that following inadequacies are inherent in these procedures:

1. Nodes should not be on reflection planes or axis of symmetries.

2. One has to use a number of global co-ordinate systems whose relative orientations depend upon the kind of the symmetries used.

3. Only parts of the total symmetries can be used by these method and one has every reason to speculate about getting more informations by using all the symmetries or symmetry elements.

4. In situations where one can use more than one sub-set but one sub-set at a time of the set of symmetry elements, it is not clear as to which sub-set will yield better results (i.e example 1 of section 3.3).

5. Since one can not use all symmetries at a time, one can not know the results of increasing or decreasing the symmetry elements.

Looking at these difficulties one has to think for some sound algebraic approach which could take care of all symmetry elements at a stretch and could avoid all the difficulties. Fortunately such an algebraic approach has been developed by Mathematicians and is known as "Group Theory". The next chapters will deal with this algebraic approach.

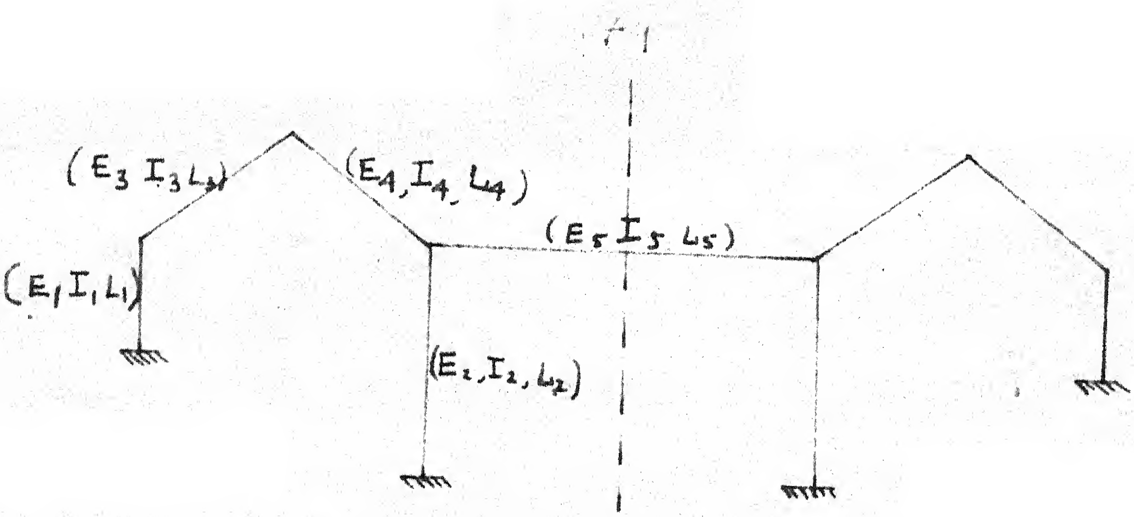


FIG. 3.1 (a)

REFLECTION SYMMETRIC FRAME

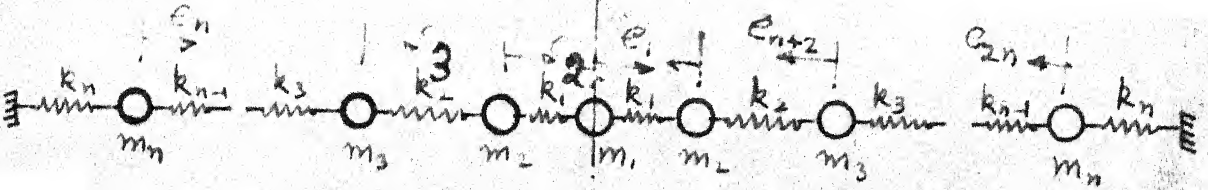


FIG. 3.1 (b)

REFLECTION SYMMETRIC SPRING MASS SYSTEM.
$e_1, e_2 \cdots e_{2n}$ ARE BASIS VECTORS.

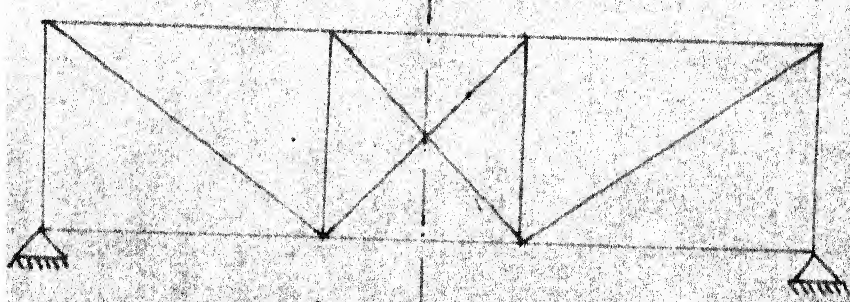


FIG. 3.1 (C)

REFLECTION SYMMETRIC PLANER TRUSS.

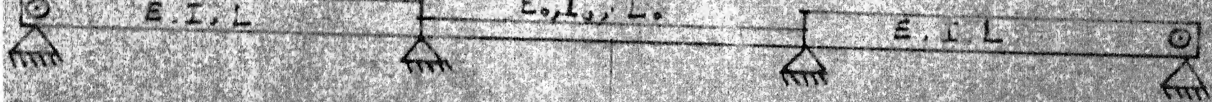


FIG. 3.1 (d)

REFLECTION PLANE : σ

REFLECTION SYMMETRIC CONTINUOUS BEAM.

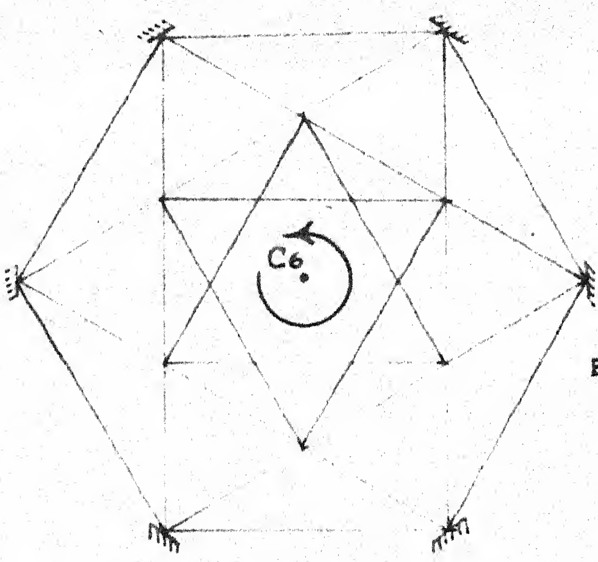


Fig. 3·2 (a).

GRID WORK WITH SYMMETRY ELEMENTS E, $C_6, C_6^2 \dots C_6^5, \sigma_1, \sigma_2 \dots \sigma_6$

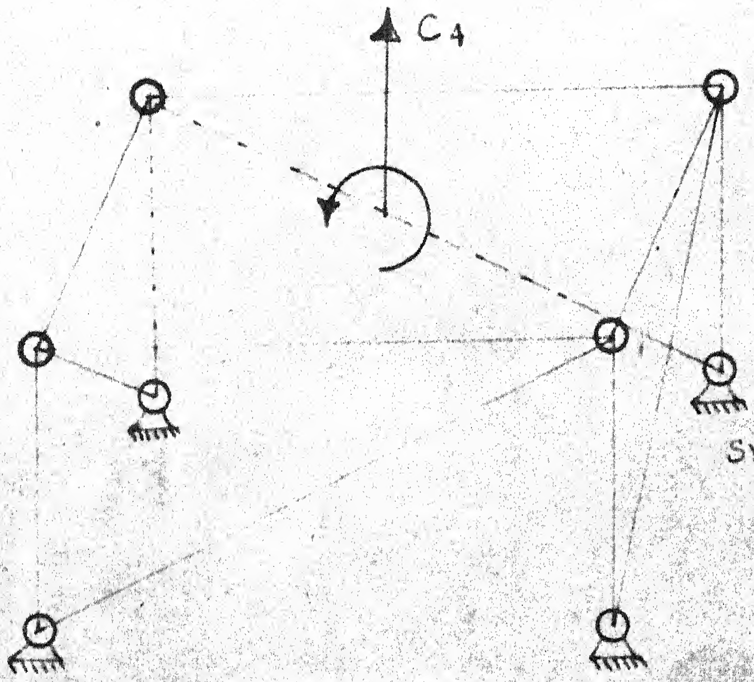


Fig. 3·2 (b)

SPACE TRUSS WITH SYMMETRY E, C_4, C_4^2, C_4^3

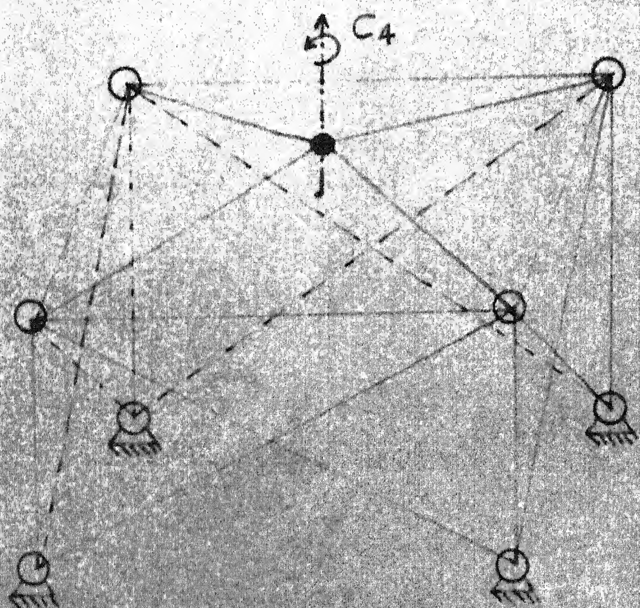


Fig. 3·2 (c)

SPACE TRUSS WITH SYMMETRY ELEMENTS E, C_3, C_3^1, C_3^3, $\sigma_1 \dots \sigma_4$

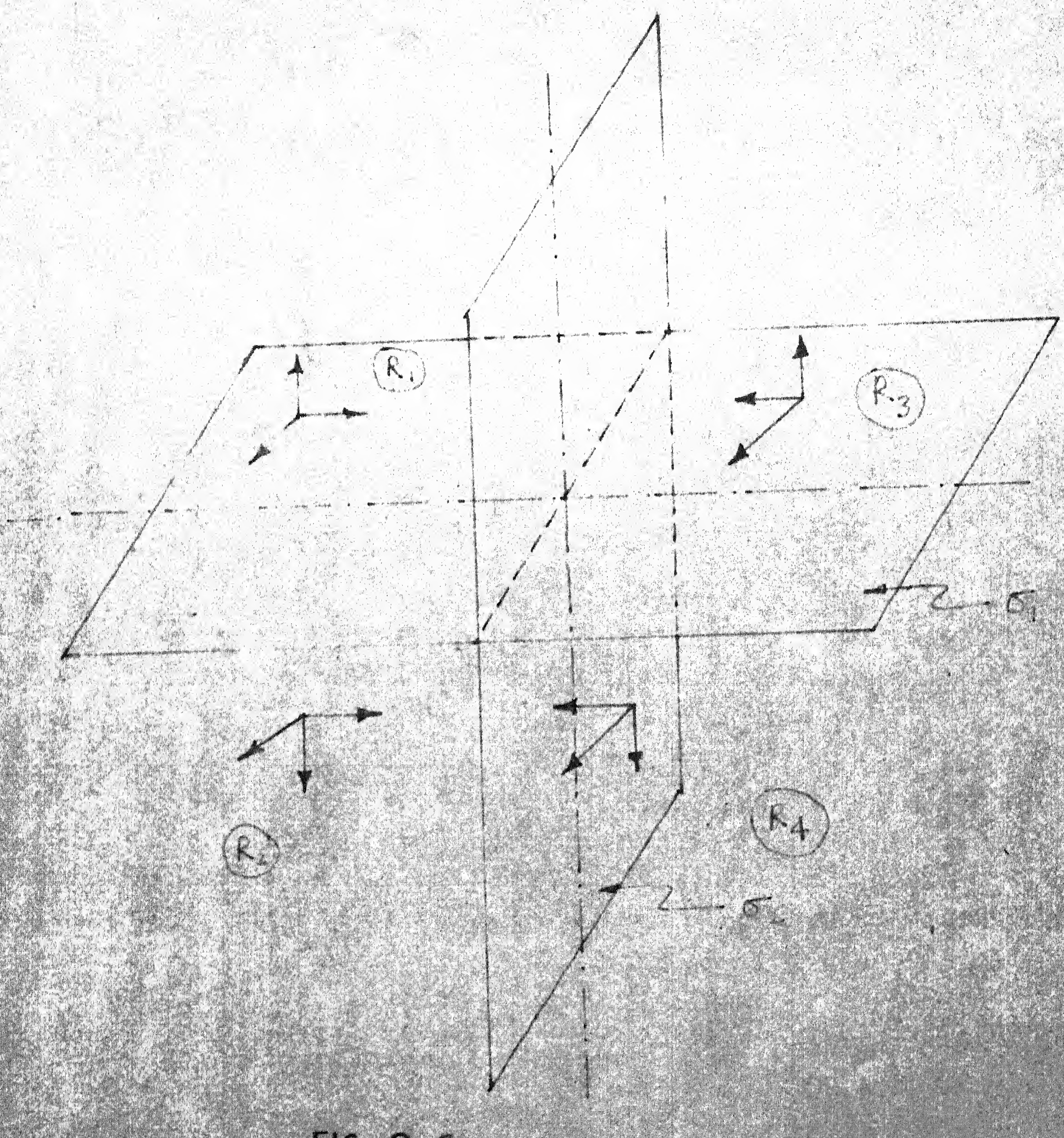


FIG. 3·6

ANY GENERAL DOUBLE REFLECTION SYMMERIC SYSTEM. R_1, R_2, R_3 AND R_4 ARE THE MIRROR SYMETRIC REGIONS OF THE STRUCTURAL SYSTEMS.

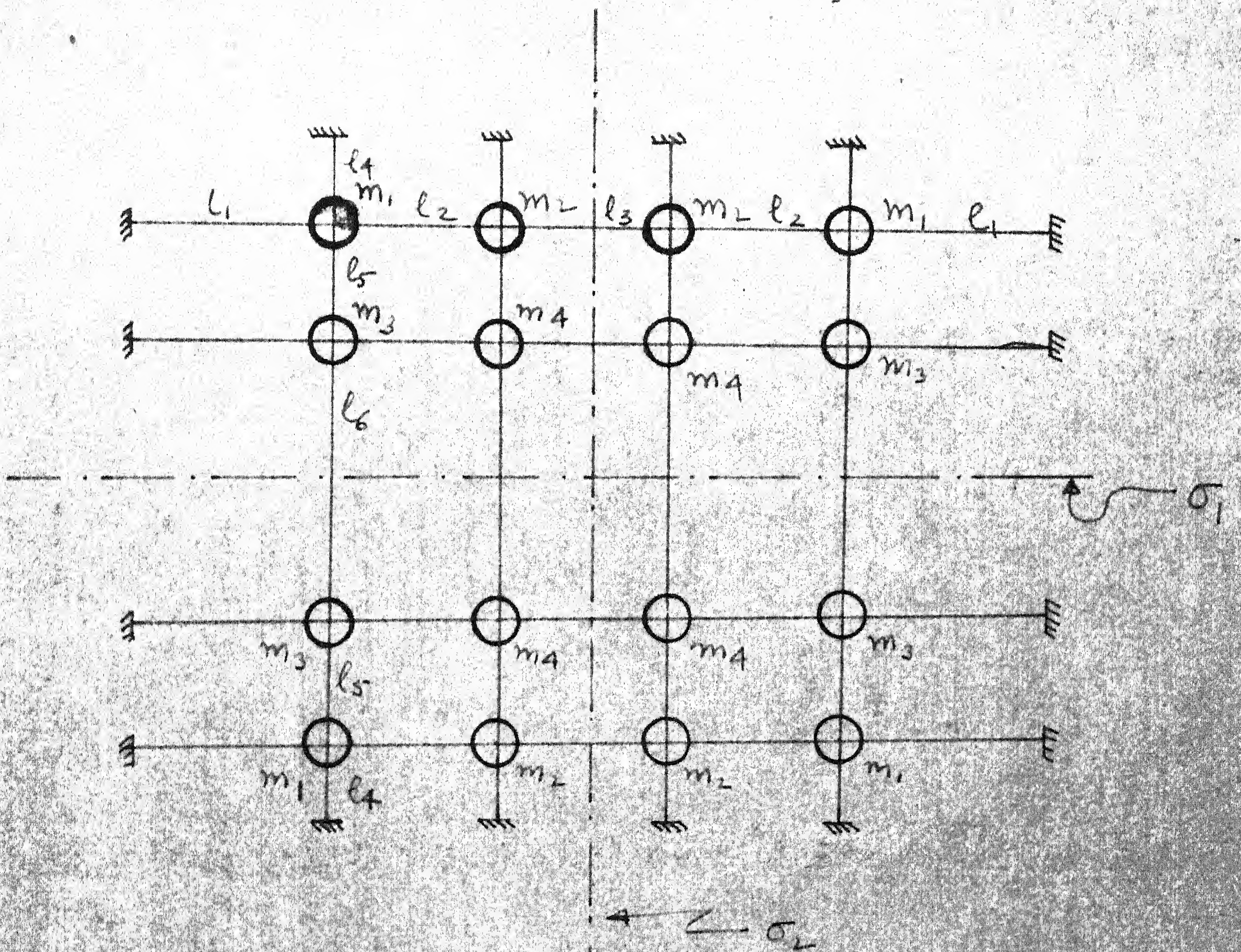


FIG. 3·8

THE 4X4 DOUBLE REFLECTION SYMMETRIC CABLE NETWORK. TENCION IN ALL THE CABLE IS 'T'. THE SYMMETRY ELEMENTS ARE, E, σ_1, σ_2 AND C.

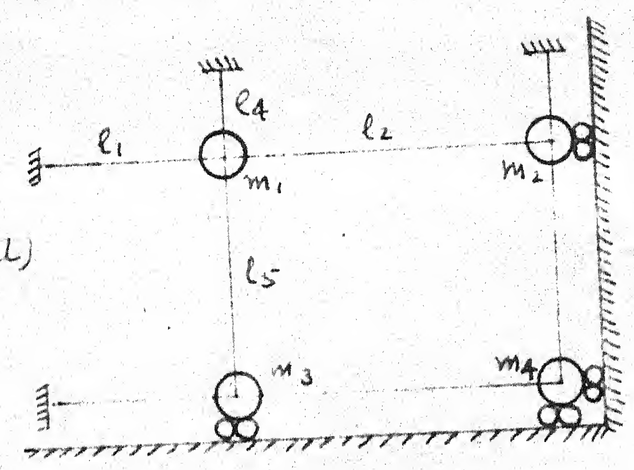


FIG. 3·8 (a)

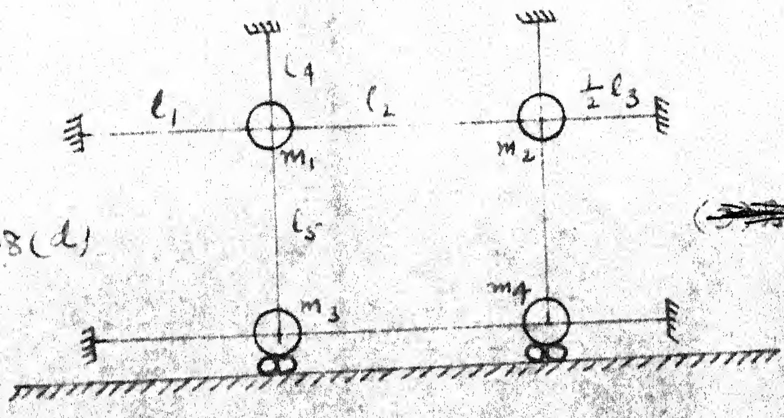


FIG 3·8(d)

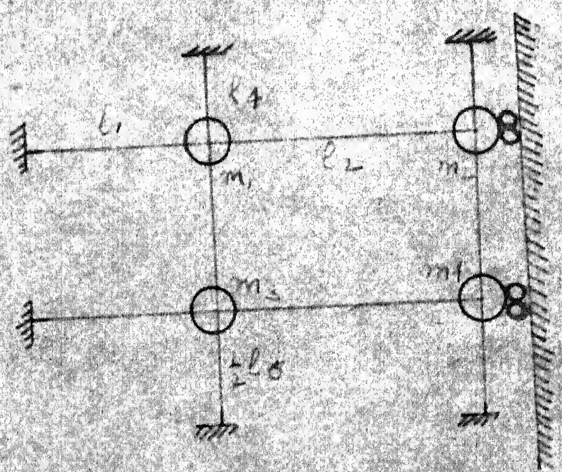


FIG 3·8 (b)

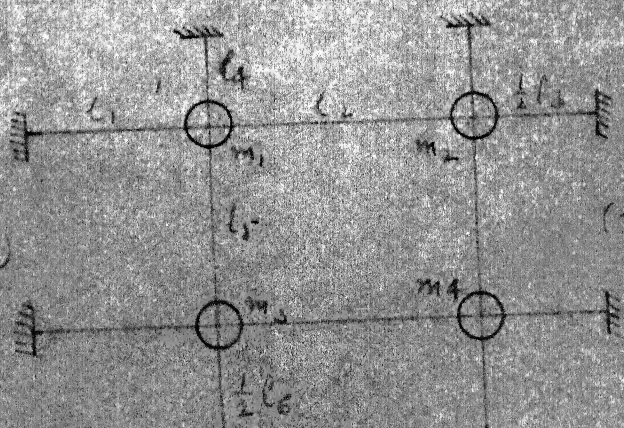


FIG 3·8 (c)

SUB-SYSTEMS OF THE SYSTEM SHOWN IN FIG 3·8

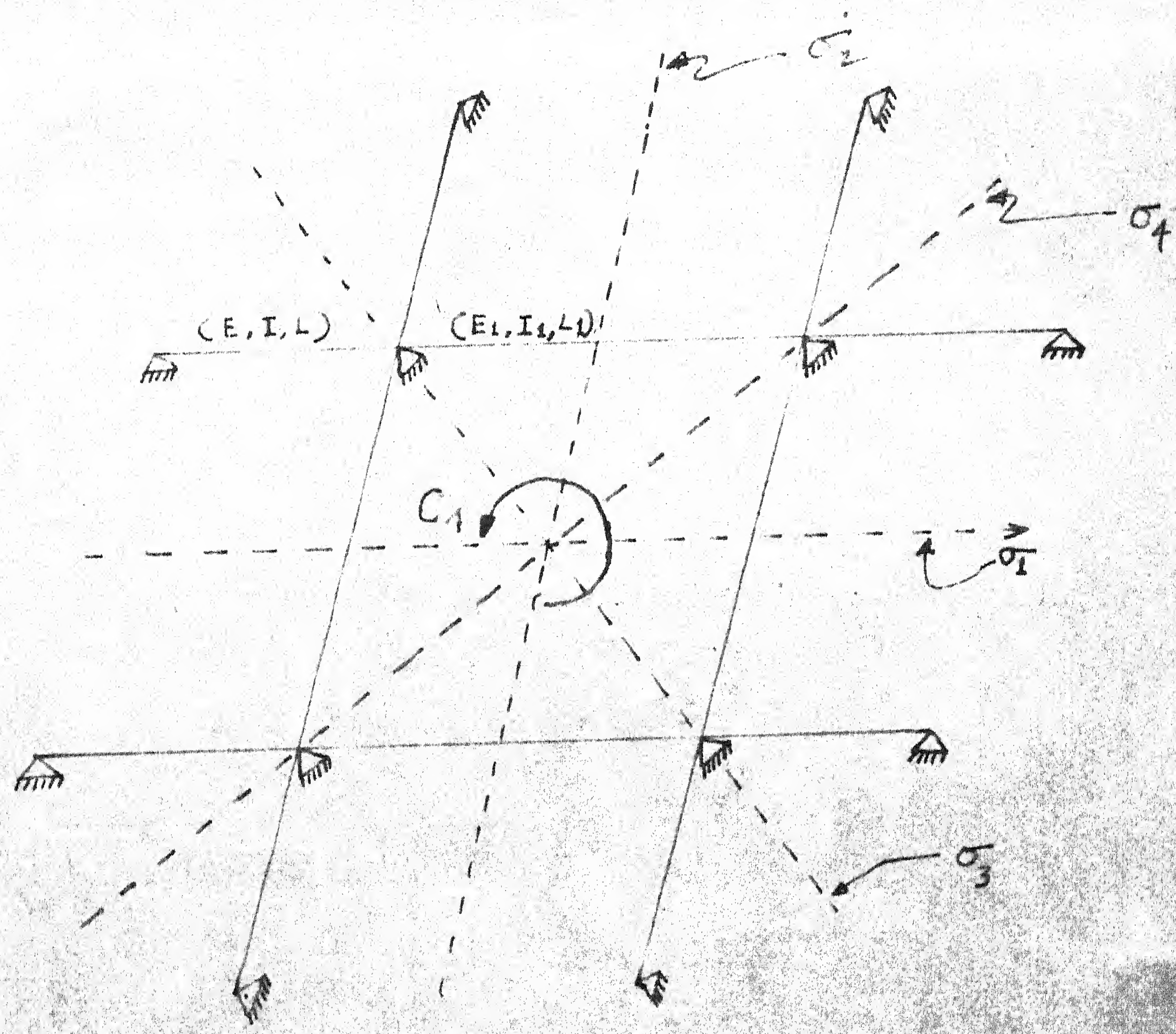


Fig. 3·9

2X2 SQUARE ORTHOGONAL SIMPLY SUPPORTED GRID WORK WITH SYMMETRY ELEMENTS, E, C_4, C_4^2, C_4^3, σ_1, σ_2, σ_3, σ_4

NOTE 1. WHILE USING GROUP THEORY REPLACE $\sigma_1 \rightarrow \sigma_2$, $\sigma_2 \rightarrow \sigma_4$, $\sigma_3 \rightarrow \sigma_4$, $\sigma_4 \rightarrow \sigma_1$

NOTE 2. FOR EXAMPES 3 AND 1 OF SECTION 3·2 AND 3·3 RESPECTIVELY, $E_1 = E$, $I_1 = I$, $L_1 = L = l$ AND THE FOUR DIFF. GLOBAL CO-ORDINATE SYSTEMS AND NUMBERINGS SHOUD BE DONE AS REQUIRED BY SECTION 3·2 AND 3·3 RESPECTIVELY.

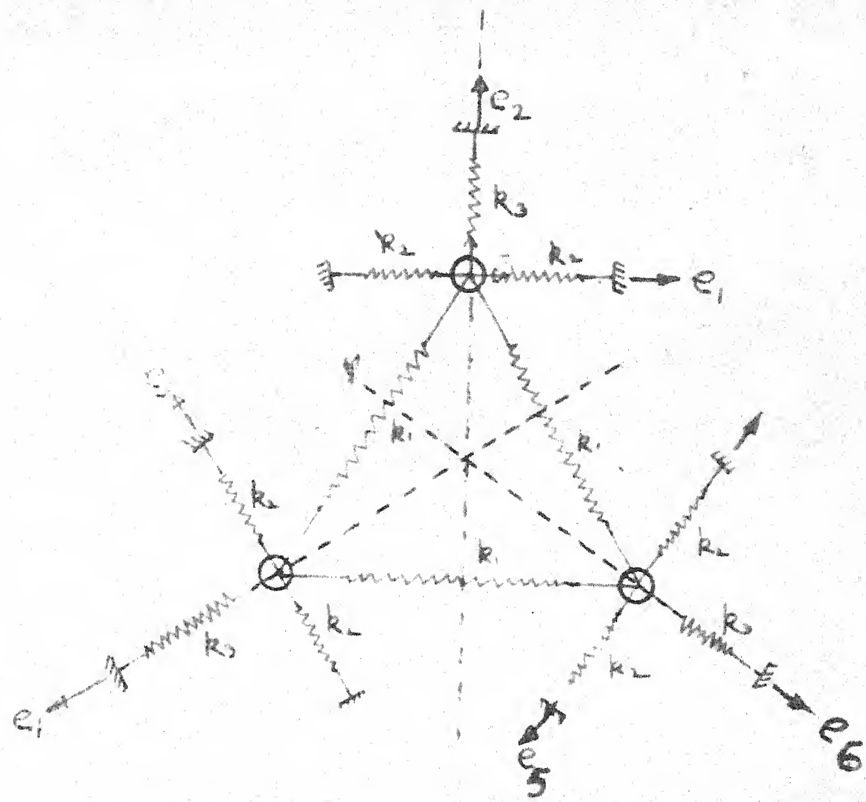


FIGURE. 3.10 (a) SPRING MASS SYSTEM WITH SYMMETRY ELEMENTS $E, C_3, C_3^2, \sigma_1, \sigma_2, \sigma_3$; e_1 ------ e_6 ARE BASIS VECTORS.

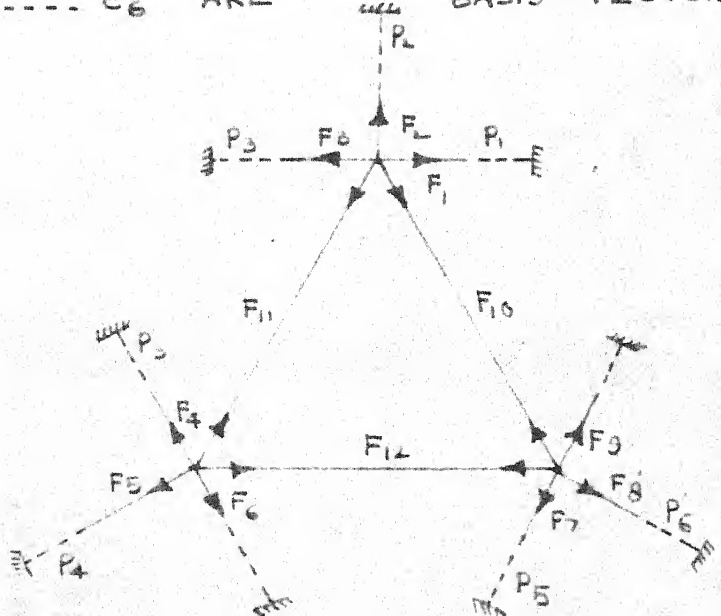


FIGURE 3.10 (b) THE CO-ORDINATE SYSTEMS CHOOSEN FOR THE EXAMPLE - 2 OF SECTION 3.3.

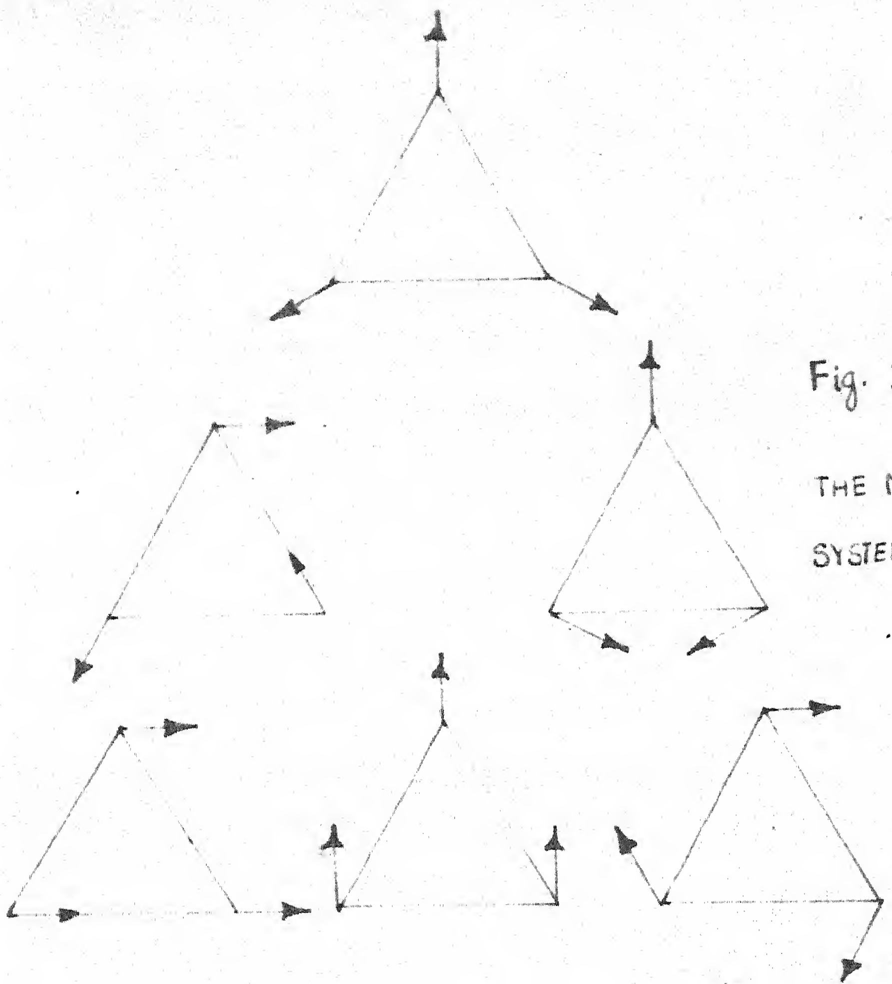


Fig. 3·10 (c)

THE MODE SHAPES OF THE SYSTEM SHOWN IN FIG. 3·10(a)

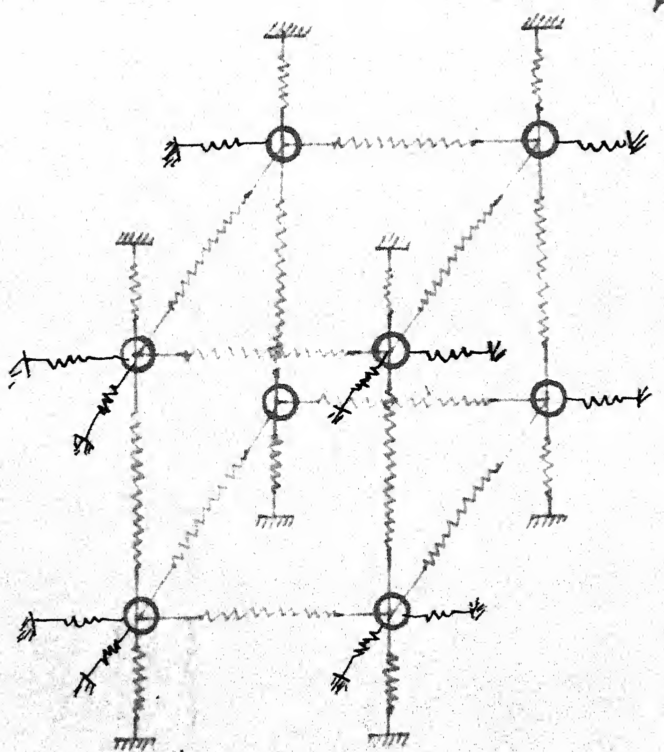


Fig. 3·11

CHAPTER - 4

GROUP THEORY

THE MATHEMATICS OF SYMMETRY

The origin of group concepts is very old. Egyptian artists, as pointed in the "introduction" for example, made use of it in the development of their geometric designs. However, the first systematic use of the concept is due to the young student antiroylist, Evariste Galois (1811-1832) who, recognizing the inevitable outcome of an arranged "duel of honor", wrote down the basic concepts of group theory in the few hours of remaining to him the day and night before his demise. Galois used this concept to simplify the solutions of algebraic equations by exploiting the symmetry properties of these equations. With these ideas later it was eassy to prove the impossibility of finding the solution of the general algebraic equations of 5-th or higher degree in terms of surds, for which mathematicians broke their heads for decades. How it helps in solving the algebraic equations can be found from excellent books due to Klein (46), Burnside (47), Lieber (48).

Just like, a one line Newton's laws, a one sentance Schrodinger's equation etc. cover volumes and volumes, the four postutates of 'Group Theory' (which have now been

brought to two only) also cover volumes and volumes. Infact this is one of the hall-mark of every elegant theory. The four postulates of group theory will be physically explained in the next section after they are stated.

In what follows, various results, which will either be directly applicable or will enable one to derive all the results wherever needed, will be just-quoted with proper illustrations. These results have been screened out from references (49 - 55) and various proofs etc. can be found from these.

4.1 GROUP THEORETIC DEFINITIONS, TERMINOLOGY AND NOTATIONS:

In the II Chapter a language for describing symmetry of any structural system was developed interms of what has been called symmetry elements and symmetry operations. It was found that a set of symmetry elements describe the symmetry of any structural system. However there seems to be one more step to go ahead (Of-course a very important step) which defines symmetry of structural system much more precisely and elegantly than the sets of symmetry elements. For this purpose the following sub-section is needed.

4.1.1. Concept of Groups:

DEFINITION: A 'group' is a set of distinct entities of any sort with a binary multiplication law such that:

(i) multiplication is closed

(ii) multiplication is associative

(iii) there exists an identity element in the set

(iv) every element has an inverse relative to the identity in (iii).

Putting these in more concrete terms, let $G = \{ g_1, g_2, g_3 - - - - - g_m - - - - \}$ be a set. This set forms a group if,

(i) for every g_i and $g_j \in G$ there exist a $g_k \in G$ such that $g_i g_j = g_k$; where binary operation is written as $g_i \; g_j$.

(ii) for all $g_i, g_j, g_k \in G$, $g_i(g_j \; g_k) = (g_i \; g_j)g_k = g_i g_j g_k$

(iii) there exist one and only one g_i which is identity E w.r.t. above binary multiplication.

(iv) for every $g_i \in G$, there exist one and only one element g_k such that, $g_i g_k = g_k \; g_i$ = identity. Then g_k is called inverse of g_i and vice-versa.

Note that the binary multiplication can be any thing which defines a function over all the ordered pairs of G and written as $g_i g_j$, e.g. if g_i are numbers the binary operations can be addition, multiplication etc., if g_i are operators then binary operation is application of one operator after the other, if g_i are nx n non-singular matrices then the binary operation canbe matrix multiplication

or addition or even direct sums or direct products etc. and if g_i are our dresses then the binary operation can be thought of putting on them one after the other. (e.g. shoes after the socks etc.).

To fix up the ideas following examples of groups are given.

1. Groups of Integers under addition:

$$G = \{ \ldots\ldots - n \ldots\ldots -2, -1, 0, 1, 2, \ldots\ldots n \ldots \}$$

It can be very easily seen that G satisfies all the postulates (i), (ii), (iii) and (iv). The identity is = 0, the inverse of n is -n etc.

2. Groups of n x n non-singular matrices under the binary operation of matrix multiplications.

$$G = \{ M_1, M_2 \ldots\ldots\ldots M_r \ldots\ldots\ldots \}$$

It can be easily seen that G satisfies all the postulates needed for a group.

3. Set of matrices $T_1\ T_2 \ldots\ldots T_m = I$ or $A_1, A_2 \ldots A_m = I$ of section 3.3, as shown there it self, form groups.

<u>DEFINITION:</u> A group is said to be finite if it has finite number of elements. The order of a finite group is the number of elements in it.

Thus the groups of example 1 and 2 are infinite groups whereas the groups of example 3 are finite and of

order 'm'. For a finite group all the postulates can be satisfied if the elements of a set form a group multiplication table known as Cayley Table for a group.

Consider now the structural systems shown in Figs. 2.9, 2.10, 3.1(a), 3.1(b) 3.4 and 3.5. All of them have two symmetry elements E and σ. It has been seen in the previous chapters that $\sigma^2 = E$ and $\sigma E = E\sigma = \sigma$.

Thus the set $G = \{E, \sigma\}$ forms a group whose multiplication can be given as the table 4.1(a). The above systems may have only E and C_2 where C_2 is a rotation by 180° i.e. the system coincides with itself under a rotation by 180° and not under σ. The set $G = \{E, C_2\}$ also forms a group whose multiplication law is given in Table 4.1(b). Similarly the structural systems shown in Figs. 2.11, 3.8 and 3.2(a), (b) and (c) have got the symmetry elements E, σ_1, σ_2 and C_2. The set $G = \{E, \sigma_1, \sigma_2, C_2\}$ forms a group can be easily seen from the Table 4.2 which is another way of putting the mnemonic of example-2 of section 2.1.2. The m-fold cyclically symmetric structural systems have symmetry elements E, C_m, C_m^2 ...... C_m^{m-1}. The set $G = \{E, C_m, C_m^2 \ldots C_m^{m-1}\}$ also forms a group because $C_m^r \; C_m^t = C_m^{r+t}$ is also one of the element from the set and the inverse of C_m^r is C_m^{m-r} etc.

The structural system shown in Figs. 2.1, 2.2, 2.12,

TABLE 4.1(a)

MULTIPLICATION TABLE OF GROUP $\{E, \sigma\}$

The Group C_{1h}

	E	σ
E	E	σ
σ	σ	E

TABLE 4.1(b)

MULTIPLICATION TABLE OF GROUP $\{E, C_2\}$

The Group C_2

	E	C_2
E	E	C_2
C_2	C_2	E

TABLE 4.2

GROUP MULTIPLICATION TABLE OF GROUP $G = \{E, \sigma_1, \sigma_2, C_2\}$

The group C_{2v}

	E	σ_1	σ_2	C_2
E	E	σ_1	σ_2	C_2
σ_1	σ_1	E	C_2	σ_2
σ_2	σ_2	C_2	E	σ_1
C_2	C_2	σ_2	σ_1	E

TABLE 4.3

The Group C_{3v}

THE MULTIPLICATION TABLE OF GROUP $\{E,\ C_3, C_3^2,\ \sigma_v^a, \sigma_v^b, \sigma_v^c\}$

	E	C_3	C_3^2	σ_v^a	σ_v^b	σ_v^c
E	E	C_3	C_3^2	σ_v^a	σ_v^b	σ_v^c
C_3	C_3	C_3^2	E	σ_v^c	σ_v^a	σ_v^b
C_3^2	C_3^2	E	C_3	σ_v^b	σ_v^c	σ_v^a
σ_v^a	σ_v^a	σ_v^b	σ_v^c	E	C_3	C_3^2
σ_v^b	σ_v^b	σ_v^c	σ_v^a	C_3^2	E	C_3
σ_v^c	σ_v^c	σ_v^a	σ_v^b	C_3	C_3^2	E

TABLE 4.4

The MULTIPLICATION TABLE OF GROUP $\{E,\ C_4, C_4^2,\ C_4^3,\ \sigma_1,\ \sigma_2,\ \sigma_3,\ \sigma_4\}$

The group C_{4v}

	E	C_4	C_4^2	C_4^3	σ_1	σ_2	σ_3	σ_4
E	E	C_4	C_4^2	C_4^3	σ_1	σ_2	σ_3	σ_4
C_4	C_4	C_4^2	C_4^3	E	σ_2	σ_3	σ_4	σ_1
C_4^2	C_4^2	C_4^3	E	C_4	σ_3	σ_4	σ_1	σ_2
C_4^3	C_4^3	E	C_4	C_4^2	σ_4	σ_1	σ_2	σ_3
σ_1	σ_1	σ_4	σ_3	σ_2	E	C_4^3	C_4^2	C_4
σ_2	σ_2	σ_1	σ_4	σ_3	C_4	E	C_4^3	C_4^2
σ_3	σ_3	σ_2	σ_1	σ_4	C_4^2	C_4	E	C_4^3
σ_4	σ_4	σ_3	σ_2	σ_1	C_4^3	C_4^2	C_4	E

TABLE 4.5

THE GROUP MULTIPLICATION TABLE OF GROUP $\{E,\ C_6,\ C_6^2 \ldots\ C_6^5, \sigma_1 \ldots \sigma_6$

The Group C_{6v}

	E	C_6	C_6^2	C_6^3	C_6^4	C_6^5	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6
	E	C_6	C_6^2	C_6^3	C_6^4	C_6^5	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6
$_6$	C_6	C_6^2	C_6^3	C_6^4	C_6^5	E	σ_2	σ_3	σ_4	σ_5	σ_6	σ_1
$^2_6$	C_6^2	C_6^3	C_6^4	C_6^5	E	C_6	σ_3	σ_4	σ_5	σ_6	σ_1	σ_2
$^3_6$	C_6^3	C_6^4	C_6^5	E	C_6	C_6^2	σ_4	σ_5	σ_6	σ_1	σ_2	σ_3
$^4_6$	C_6^4	C_6^5	E	C_6	C_6^2	C_6^3	σ_5	σ_6	σ_1	σ_2	σ_3	σ_4
$^5_6$	C_6^5	E	C_6	C_6^2	C_6^3	C_6^4	σ_6	σ_1	σ_2	σ_3	σ_4	σ_5
$_1$	σ_1	σ_6	σ_5	σ_4	σ_3	σ_2	E	C_6^5	C_6^4	C_6^3	C_6^2	C_6
$_2$	σ_2	σ_1	σ_6	σ_5	σ_4	σ_3	C_6	E	C_6^5	C_6^4	C_6^3	C_6^2
$_3$	σ_3	σ_2	σ_1	σ_6	σ_5	σ_4	C_6^2	C_6	E	C_6^5	C_6^4	C_6^3
$_4$	σ_4	σ_3	σ_2	σ_1	σ_6	σ_5	C_6^3	C_6^2	C_6	E	C_6^5	C_6^4
$_5$	σ_5	σ_4	σ_3	σ_2	σ_1	σ_6	C_6^4	C_6^3	C_6^2	C_6	E	C_6^5
$_6$	σ_6	σ_5	σ_4	σ_3	σ_2	σ_1	C_6^5	C_6^4	C_6^3	C_6^2	C_6	E

TABLE 4.6

THE MULTIPLICATION TABLE OF THE GROUP -

$\{E,\ C_3,\ C_3^2,\ \sigma_v^a, \sigma_v^b, \sigma_v^c,\ \sigma_h\ S_3,\ S_3^2,\ C_2^a,\ C_2^b,\ C_2^c\}$

The Group D_{3h}

	E	C_3	C_3^2	σ_v^a	σ_v^b	σ_v^c	σ_h	S_3	S_3^2	C_2^a	C_2^b	C_2^c
E	E	C_3	C_3^2	σ_v^a	σ_v^b	σ_v^c	σ_h	S_3	S_3^2	C_2^a	C_2^b	C_2^c
C_3	C_3	C_3^2	E	σ_v^c	σ_v^a	σ_v^b	S_3	S_3^2	σ_h	C_2^c	C_2^a	C_2^6
C_3^2	C_3^2	E	C_3	σ_v^b	σ_v^c	σ_v^a	S_3^2	σ_h	S_3	C_2^b	C_2^c	C_2^a
σ_v^a	σ_v^a	σ_v^b	σ_v^c	E	C_3	C_3^2	C_2^a	C_2^b	C_2^c	σ_h	S_3	S_3^2
σ_v^b	σ_v^b	σ_v^c	σ_v^a	C_3^2	E	C_3	C_2^b	C_2^c	C_2^a	S_3^2	σ_h	S_3
σ_v^c	σ_v^c	σ_v^a	σ_v^b	C_3	C_3^2	E	C_2^c	C_2^a	C_2^b	S_3	S_3^2	σ_h
σ_h	σ_h	S_3	S_3^2	C_2^a	C_2^b	C_2^c	E	C_3	C_3^2	σ_v^a	σ_v^b	σ_v^c
S_3	S_3	S_3^2	σ_h	C_2^c	C_2^a	C_2^b	C_3	C_3^2	E	σ_v^c	σ_v^a	σ_v^b
S_3^2	S_3^2	σ_h	S_3	C_2^a	C_2^c	C_2^a	C_3^2	E	C_3	σ_v^b	σ_v^c	σ_v^a
C_2^a	C_2^a	C_2^b	C_2^c	σ_h	S_3	S_3^2	σ_v^a	σ_v^b	σ_v^c	E	C_3	C_3^2
C_2^b	C_2^b	C_2^c	C_2^a	S_3^2	σ_h	S_3	σ_v^b	σ_v^c	σ_v^a	C_3^2	E	C_3
C_2^c	C_2^c	C_2^a	C_2^b	S_3	S_3^2	σ_h	σ_v^c	σ_v^a	σ_v^b	C_3	C_3^2	E

The thickly bordared parts are sub-groups C_3 and C_{3v} of th group D_{3h}.

2.13 and 3.10 (a) have got the following set of symmetry elements.

$$G = \left\{ E,\ C_3,\ C_3^2,\ \sigma_u^a,\ \sigma_u^b,\ \sigma_u^c \right\}$$

This set also forms a group can be seen from the Table 4.3 which follows from the mnemonic of example -3 of section 2.1.2.

The system shown in Fig. 3.9 has symmetry elements $E,\ C_4,\ C_4^2,\ C_4^3,\ \sigma_1,\ \sigma_2,\ \sigma_3,\ \sigma_4$ which also form a group as shown by the group multiplication Table 4.4. The set of symmetry elements

$$G = \left\{ E,\ C_6,\ C_6^2,\ C_6^3,\ C_6^4,\ C_6^5,\ \sigma_1,\ \sigma_2,\ \sigma_3,\ \sigma_4,\ \sigma_5,\ \sigma_6 \right\}$$

which correspond to the structural systems shown in Figs. 2.14 and 2.15 also forms a group can be seen from the group multiplication table 4.5 which follows from the mnemonic scheme of example-4 of section 2.1.2.

Similarly the symmetry elements of structural system of Fig. 2.16 also form a group which is seen from the group multiplication Table 4.6 which follows from the mnemonic scheme of example-5 of section 2.1.2.

From these examples one can observe that the symmetry of structural systems can be described in terms of groups of symmetry elements. The following conjecture can be made now:

<u>CONJECTURE:</u> <u>The symmetry elements of a structural system form a group.</u>

<u>Note:</u> Since under all the aforementioned symmetry operations there is at least one point which remains fixed, the groups of such symmetry elements are called point groups. Next sub-section will categorise all possible point groups.

<u>DEFINITION:</u> A group is called Abelian if and only if for every g_i, $g_j \in G$ $g_i g_j = g_j g_i$.

4.1.2 <u>The Symmetry Point Groups of Structural Systems:</u>

Following are the possible point groups which can describe the symmetry of any possible structure.

C_1 The point group containing only identity E. i.e. this point group correspond to the structures with no symmetry.

C_n The point group containing, E, C_n, C_n^2 ..... C_n^{n-1} i.e. this point group correspond to the structural systems with cyclic symmetry, e.g. reticular dome.

C_{nv} The point group consisting of E, C_n, C_n^2, C_n^3 .... C_n^{n-1} and n-reflection planes passing through the axis C_n (viz. σ_1, σ_2, ... σ_n).

C_{nh} The point group consisting of all elements of C_n and σ_h, $\sigma_h C_n = S_n$ $\sigma_h C_n^2 = S_n^2$ .... $\sigma_h C_n^{n-1} = S_n^{n-1}$

where σ_h is a reflection plane perpendicular to principal axis C_n.

D_n; The point group consisting of all elements of C_n and 'n' two-fold axes perpendicular to the principal axis C_n i.e. with the elements E, $C_n \ldots C_n^{n-1}$, $C_2^{a_1}$, $C_2^{a_2}$ ...... $C_2^{a_n}$.

D_{nh}: The point group consisting of elements of D_n and σ_h, $\sigma_h C_n = S_n$, $\sigma_h C_n^2 = S_n^2$, ... $\sigma_h C_n^{n-1} = S_n^{n-1}$ and n-reflection planes through axis C_n.

D_{nd}: Consisting of D_n and a set of planes passing through the principal axis C_n and bisecting the angles between two-fold axes.

S_n: The point group with E, S_n S_n^2 .... S_n^{n-1} for n=even, (for n = odd this point group is nothing but C_{nh} because $S_n^k = \sigma_h C_n^k$).

T: The point group corresponding to that structural systems which posses all those rotation axes (please see II-Chapter) which are possessed by the tetrahedral frame shown in Fig. 2.3.

T_d: The point group containing all the elements of the tetrahedral frame of Fig. 2.3.

O: The point group consisting of all the ~~rotation axes~~ rotation axes of the system shown in Fig. 3.11.

O_h: The point group consisting of all those symmetry elements possessed by the system shown in Fig. 3.11 etc.

But engineering structures usually have C_1, C_n, C_{nv} and C_{nh} point groups. The structural systems with point groups D_n, D_{nh}, D_{nd}, S_n, T, T_d, O and O_h etc. are very few. However, noteworthy is the point THAT ALL POSSIBLE STRUCTURAL SYSTEMS WILL HAVE ONE OF THE SYMMETRY POINT GROUP from the point groups C_n, C_{nh}, C_{nv}, D_n, D_{nh}, D_{nd}, S_n, T, T_d, O and O_h etc. where n=1,2,3,,........

The point groups C_n and C_{nh} are always Abelian. C_{2v} is also Abelian. Symmetries corresponding to an Abelian point group can be used to simplify the structural problems more easily then that corresponding to non-Abelian groups. Examples of the symmetry groups C_{nh}, C_{nv}, and D_{nh} are given in Tables 4.1 - 4.6 and thus one can see what kinds of structural system correspond to these groups.

4.1.3 Sub-groups, Cosets and Classes of Point Groups:

DEFINITION: "If a subset of a group is also a group, then that subset is called a subgroup of the group". e.g. the point group shown in Table 4.2 has three subgroups, $\{E, \sigma_1\}$, $\{E\ \sigma_2\}$ and $\{E, C_2\}$, the point group shown in Table 4.3 has one subgroup $\{E, C_3, C_3^2\}$, the point group of Table 4.4 has seven subgroups $\{E, \sigma_1\}$, $\{E, \sigma_2\}$, $\{E, \sigma_3\}$, $\{E, \sigma_4\}$, $\{E, \sigma_1, \sigma_3, C_4^2\}$, $\{E, \sigma_2, \sigma_4, C_4^2\}$ and $\{E, C_4\ C_4^2, C_4^3\}$ and the point group of table 4.6 has three subgroups $\{E, C_3, C_3^2\}$, $\{E, C_3, C_3^2\ \sigma_v^a, \sigma_v^b, \sigma_v^c\}$ and $\{E, C_3, C_3^2, \sigma_h, S_3, S_3^2\}$.

Note that every group is a sub-group by itself and E is subgroup of every group but these trivial sub-groups are of no interest. There are various special types of subgroups of a group which are called INVARIANT SUBGROUPS, COMUTATOR SUBGROUP etc. and are very important mathematically. An important theorem due to Lagrangs states that the order of a subgroup of a group is a divisior of the order of the group (i.e. if h = order of a subgroup and g= order of the group then g = nh, n = some integre). This theorem is proved with the help of 'Cosets' which are subsets of the group obtained from left or right multiplications of various subgroups of the group by the elements not belonging to the subgroup and are of only mathematical interest.

Another important way of deviding a group into different subsets is by defining a class structure on the group.

<u>DEFINITION</u>: "Let P and Q be two members of a group. They are said to belong to the same "Class" if they are related by the equation,

$$Q = X^{-1} P X$$

where X is any member of the group including P and Q themselves".

For example, the classes of the group (E, C_3, C_3^2) are(E) ,(C_3), (C_3^2) i.e. every element of the group C_3 is a class by itself; the classes of the group C_{3v} (Table 4.3) are(E), (C_3, C_3^2) and (σ_v^a , σ_v^b , σ_v^c) ; the

classes of C_{4v} (Table 4.4) are (E) , (C_4^2), (C_4, C_4^3), (σ_1, σ_3) and (σ_2, σ_4); the classes of C_{2v} (Table 4.2) are (E), (σ_1), (σ_2) and (C_2) and the classes of D_{3h} (Table 4.6) are (E), (C_3, C_3^2), (σ_v^a, σ_v^b, σ_v^c), (σ_h), (S_3,S_3^2) and (C_2^a, C_2^b, C_2^c). Let us consider a group G of order g. Let K_1, K_2 ..... K_r are its r classes (Note that every element of a group belongs to only one class). Let the class K_i has r_i members then

$$\sum_{i=1}^{r} r_i = g$$

and if g = odd integer then g-r must be divisible by 16. The following results are also of interest:

(i) EVERY ELEMENT OF AN ABELIAN GROUP IS A CLASS BY ITSELF (A very very important result as will be seen later).

(ii) Every element of a class K_i has same order (where the 'order' of an element g_i is an integer m such that $g_i^m = E$) and this 'order' is a **divisor** of g/r_i. This enables one to get all the members of a class atonce.

Another set of results about the'class sum' and class product (set-theoretic sum) and useful in calculating some class function to be defined latter are following:

$$K_i \quad K_j = \sum_{k=1}^{r} h_{ij,k} K_k$$ where $h_{ij,k}$ are called

class multiplication co-efficients with following properties: (Note: K_i stands both for classes and class sums).

(i) $h_{ij,k} = h_{ji,k}$

(ii) $\sum_{k=1}^{r} h_{ij,k} \; h_{kl,m} = \sum_{k=1}^{r} h_{jl,k} \; h_{ik,m}$

(iii) $r_i \; r_j = \sum_{k=1}^{r} h_{ij,k} \; r_k$ etc.

As an illustration to this, consider the group C_{3v} (Table 4.3).

The classes are, $K_1 = (E)$, $K_2 = (C_3, C_3^2)$ and $K_3 = (\sigma_v^a, \sigma_v^b, \sigma_v^c)$ which also can be written

as $K_1 = E$

$K_2 = C_3 + C_3^2$ (set theoretic sum)

$K_3 = \sigma_v^a + \sigma_v^b + \sigma_v^c$

Then $K_2K_3 = (C_3 + C_3^2)(\sigma_v^a + \sigma_v^b + \sigma_v^c)$

$$= (\sigma_v^b + \sigma_v^c + \sigma_v^a)(\sigma_v^c + \sigma_v^a + \sigma_v^b) = 2K_3$$

Thus $h_{23,1} = h_{23,2} = 0$, $h_{23,3} = 2$

$h_{33,1} = h_{33,2} = 3$, $h_{33,3} = 0$

In a similar manner one can determine all the class products (One has to use the group table at every step , (Table 4.3 for this example).

An important class of a group is the commutator class of a group. The class whose every element is the commutator element of the group where by commutator element is meant the element of the form $P^{-1}Q^{-1}PQ$ for P, Q belonging to the group and "a 'commutator subgroup' of a group is the minimal subgroup which contains all the commutators of the group". The commutator subgroup is determined from the class K_i plus few more elements (as necessary for formation of a group) provided $h_{ij,j} \neq 0$ for some j.

If h_c is the order of the commutator subgroup and g is the order of the group the number (g/h_c) will be of interest latter.

4.1.4 Direct Products:

A group G is said to be the 'direct product' of subgroups H_1, H_2 .... H_m if :

(1) The elements of different subgroups commute

(2) Every element 'g' of G is expressible as:

$$g = h_1 h_2 \ldots\ldots h_m \quad \text{where } h_i \in H_i$$

Symbolically one writes,

$$G = H_1 X\ H_2 X\ H_3 X \ldots\ldots\ldots X\ H_m$$

The sub-groups H_1, H_2 ..... H_m are called 'direct factors' of G and the order of G is the product of the orders of H_1, H_2 .... H_m, (H_1, H_2 .... H_m are non-trivial subgroups).

Following are the examples of direct product groups.

1. Cyclic group $C_6 = \{E, C_6, C_6^2 \ldots\ldots C_6^5\}$ is direct product of $C_3 = \{E, C_3, C_3^2\}$ and $C_2 = \{E, C_2\}$ because

$$C_3 \times C_2 = \{E, C_3, C_3^2, C_2, C_2 C_3, C_2 C_3^2\}$$

$$= \{E, C_6^2, C_6^4, C_6^3, C_6^5, C_6\} \text{ because,}$$

C_n^m is a rotation by $\frac{2m\pi}{n}$ etc.

or $C_3 \times C_2 = \{E, C_6, C_6^2, C_6^3, C_6^4, C_6^5\}$

2. The group $C_{2v} = \{E, \sigma_1, \sigma_2, C_2\}$ of Table 4.2 is a direct product of two groups $\{E, \sigma_1\}$ and $\{E, \sigma_2\}$ i.e. $C_{2v} = \{E, \sigma_1\} \times \{E, \sigma_2\} = C_{1h} \times C_{1h}$

3. The group $C_{nh} = \{E, C_n, C_n^2, \ldots\ldots C_n^{n-1}, \sigma_h, \sigma_h C_n = S_n, \ldots\ldots\ldots \sigma_h C_n^{n-1} = S_n^{n-1}\}$ can be written as:

$$C_{nh} = \{E, \sigma_h\} \times \{E, C_n, C_n^2, \ldots\ldots C_n^{n-1}\}$$

$$= C_{1h} \times C_n$$

In fact every Abelian group is a direct product of cyclic groups whose orders are powers of primes. The concept of direct product will be seen to be useful in latter sections. To this end, one can see that infact only few elements of a group are of fundamental nature and other members are generated from these members. e.g. in the group $C_{2v} = \{E, \sigma_1, \sigma_2, C_2\}$ only σ_1 and σ_2 are fundamental because $E = \sigma_1^2 = \sigma_2^2$, $C_2 = \sigma_1 \sigma_2 = \sigma_2 \sigma_1$.

Similarly the group $C_n = \{ E, C_n, C_n^2, \ldots\ldots C_n^{n-1} \}$ are generated by one element C_n. The group C_{nh} are generated by two elements C_n and σ_h and so on. "The minimum number of elements which can generate the whole group are called generators of the group". This concept will be helpful in finding out the 'representations' etc. of a group (the work of next section).

4.1.5 Functions on a Group and Class Functions:

A 'function on a group' is a correspondence which associates a number, a vector, a matrix etc. with each element of the group such that these numbers, vectors, matrices etc. also satisfy the group postulates or group tables. The correspondence is called isomorphic if to each element of the group different numbers, vectors, matrices etc. are associated. The correspondence is called homomorphic if to more than one element of the group is associated one number, one vector/or one matrix etc. Note that the associated numbers, vectors, matrices etc. themselves form groups (isomorphic or homomorphic groups depending upon the corresponding mappings).

A 'class function' is a function on a group which associates the same value for all elements of a class of the group.

These two kinds of functions on a group give rise to the concepts of representation and characters of a group, the work of next two sections.

4.2 REPRESENTATIONS OF GROUPS:

As discussed in the sub-section 4.1.5, one can associate anything with the elements of a group such that those things also satisfy the group table. These things can be numbers, matrices, or even the elements of some other group. Of our interests are matrices only. So let us associate a set of square non-singular matrices one with each element of the group. i.e. with

g_1 the associated matrix is $D(g_1)$
g_2 the associated matrix is $D(g_2)$
- - - - - - - - - - - - - - - -
g_n the associated matrix is $D(g_n)$

where $g_1, g_2, \ldots\ldots g_n$ form a group G.

Thus for $g_i g_j = g_k$ for $g_i, g_j, g_k \in G$, corresponding $D(g_i)\,D(g_j) = D(g_k)$, $D(E) = I$ the identify matrix and for $g_k = g_i^{-1}$ corresponds the $D(g_k) = D^{-1}(g_i)$.

Thus $\{D(g_1), D(g_2) \ldots\ldots D(g_n)\}$ form a matrix group 'R' homomorphic or isomorphic to G.

"The Group R is Called Matrix Representation of G"

The dimensions of $D(g_i)$, $(i=1,2,3, \ldots .n)$ can be 1,2,3,.... and the corresponding representations will be called respectively one-dimensional, two-dimensional, three-dimensional ...... representations of the group G. Followings are the

example of representations (from now onward matrix representations will be called just representations) of various groups.

1. The representations of the group $C_{1h} = \{E, \sigma\}$ are:

(i) One dimensional trivial representation $D(E) = 1$, $D(\sigma) = 1$

(ii) One dimensional non-trivial representation $D(E) = 1, D(\sigma) = -1$

(iii) Two dimensional representations are:

$$D(E) = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \quad D(\sigma) = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \quad \text{and}$$

$$D(E) = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \quad D(\sigma) = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \quad \text{etc.}$$

(iv) Three dimensional representations are:

$$D(E) = \begin{bmatrix}1 & & 0\\ & 1 & \\ 0 & & 1\end{bmatrix}, \quad D(E) = \begin{bmatrix}1 & & 0\\ & 1 & \\ 0 & & -1\end{bmatrix} \quad \text{and}$$

$$D(E) = \begin{bmatrix}1 & & 0\\ & 1 & \\ 0 & & 1\end{bmatrix} \quad D(\sigma) = \begin{bmatrix}1 & & 0\\ & -1 & \\ 0 & & -1\end{bmatrix} \quad \text{etc.}$$

The 2n-dimensional representations are:

$$D(E) = \begin{bmatrix}I & 0\\ 0 & I\end{bmatrix}, \quad D(\sigma) = \begin{bmatrix}I & 0\\ 0 & -I\end{bmatrix}; \quad D(E) = \begin{bmatrix}I & 0\\ 0 & I\end{bmatrix},$$

$$D(\sigma) = \begin{bmatrix}0 & I\\ I & 0\end{bmatrix} \quad \text{etc. and so on.}$$

2. The reprsentations of the group C_{2v} are:

(i) One dimensional trivial representation is,

$$D(E) = D(\sigma_1) = D(\sigma_2) = D(C_2) = 1$$

(ii) One dimensional non-trivial representation is

$D(E) = D(C_2) = 1$ $D(\sigma_1) = D(\sigma_2) = -1.$

(iii) Two dimensional representations are:

$$D(E) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = D(C_2), \; D(\sigma_1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\text{and } D(E) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D(\sigma_1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$D(\sigma_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad D(C_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ etc.}$$

One of the 4n-dimensional representation is the set of matrices I, T_1, T_2 and T_1T_2 of section 3.2. and so on ....
One can see that for any group finite or infinite there are infinity of representations. Firstly there are enormous number of different representations for the same dimensional representation and secondly there can be representations of any arbitrary dimension.

It should be noted that a similarity transformation of a representation by an arbitrary matrix generates another representation of the same dimension because, if

$R = \{ D(g_1), \; D(g_2) \; ...... \; D(g_n) \}$ is a representation of G

i.e. if $g_i \, g_j = g_k \Rightarrow D(g_i) \, D(g_i) = D(g_k)$ etc., then

$R' = \{ S^{-1} \, D(g_1) \, S, \; S^{-1} \, D(g_2) \, S, \; ...... \; S^{-1} D(g_n) \, S \}$ is also a

There are c_1 blocks of $D_1\,(g_i)$ and c_2 blocks of $D_2(g_i)$ where c_1, c_2 are positive integers (including zero) such that $c_1d_1 + c_2d_2 = d$, $D_1\,(g_i)$ and $D_2(g_i)$ are the matrices of dimensions d_1 and d_2 corresponding to the irreducible representations R_1 and R_2. The $D(g_i)$ are written as $D(g_i) = c_1\,D_1(g_i) \oplus c_2\,D_2(g_i)$ (A direct sum)

... (1)

In the general case when the group has r-classes, there will be 'r' irreducible representations $R_k = \left\{ D_k(g_i) \right\}$ $(k=1,2, \ldots. r, i=1,2, \ldots.n)$ of dimensions $d_1 \ldots. d_r$, then for any representation $R = \left\{ D(g_i) \right\}$ of dimension 'd' one can generalise the eq. (1) to following.

$$D(g_i) = c_1D_1\,(g_i) \oplus c_2\,D_2(g_i) \oplus \ldots. \oplus c_rD_r(g_i) \quad \ldots (2)$$

where $c_1, c_2 \ldots.. c_r$ are the number of blocks of matrices $D_1(g_i)$, $D_2\,(g_i) \ldots\ldots D_r(g_i)$ respectively in $D(g_i)$ for $i = 1,2, \ldots.n$, on their diagonals. One must have

$$c_1\,d_1 + c_2\,d_2 + \ldots\ldots\ldots + c_r\,d_r = d \qquad \ldots \quad \ldots (3)$$

The representation $R = \left\{ D(g_i) \right\}$ in such a form is called a reduced 'representation'. This reduced reprsentation is obtained by applying some similarity transformation to unreduced representation R' of the same dimension as that of R. "There is a direct connection between the reduction of a representation of symmetry group of a structural system by similarity transformation and the reduction of stiffness matrix to block diagonal matrics of that structural system".

The ROLE OF IRREDUCIBLE REPRESENTATIONS IN GROUP THEORY IS EQUIVALENT TO THAT OF BASIS VECTORS IN VECTOR SPACES. One can get all the informations obtainable from the use of group theory through the irreducible representations of the group. How to get these irreducible representations of any group will be discussed latter.

In order to be able to illustrate the matters discussed in this section, the concept of 'character' is a must and hence the comming sub-sections will be devoted to this.

4.2.1 Characters of Representation Matrices or Group:

The character of a representation matrix is defined as the sum of its diagonal elements (the trace of the matrix). Why they are called character and not the trace will become clear very soon when it will be shown that they are same for a set of elements belonging to a class (i.e. they are class functions).

Now if $R = \{D(g_1) \quad D(g_2) \; \ldots\ldots \; D(g_n)\}$ be the d-dimensional representation of a group $G = \{g_1, g_2 \ldots\ldots g_n\}$ then character of various elements say $k_1, k_2 \ldots\ldots k_n$ are,

$k_1 = \text{tr. } D(g_1)$ where tr. stands for trace.

$k_2 = \text{tr. } D(g_2)$

- - - - - - - - -

$k_n = \text{tr. } D(g_n)$

Consider now another representation R' generated from R by similarity transformation by an arbitrary appropriate matrix S. i.e. $R' = \{ S^{-1} D(g_i) S \}$

That characters of R' and R are same, follows from the well known result of matrix theory, viz. trace (ABC) = trace (BCA) = trace (CAB) i.e. if $k'_1, k'_2 \ldots k'_n$ are characters of $R' = \{ S^{-1} D(g_i) S \}$ then,

$k'_i = tr. (S^{-1} D(g_i) S) = tr. (D(g_i)SS^{-1} = tr.D(g_i) = k_i$

Let us divide the group into distinct classes, $K_1, K_2, K_3 \ldots K_r$ where the 'Class' was defined in section 4.1.3 as follows: Two elements P and Q of the group belong to the same class K_i if there exists an element X in the group (including P and Q) such that,

$$Q = X^{-1} P X$$

The corresponding definition for representation matrices will be $D(Q) = D^{-1}(X) D(P) D(X)$

Thus characters of P and Q are same provided they belong to the same class and so the characters of a group are as many as there the classes. Let us say that character of class K_i is k_i (i=1,2, r).

There are two ways of calculating the characters of a group for a given representation. One way is to find the representation matrices corresponding to one element

per class and then to find the traces. The other shorter and more elegand method is to use what is known as Burnside theorem which follows from the concept of sums introduced earlier i.e.

$$K_i\ K_j = \sum_{k=1}^{r} h_{ij,k} K_k \tag{4}$$

Now in representation $R = \{D(g_i)\}$ one has the corresponding matrices for class sums K_i ($i=1,2, \ldots r$) say $D(K_i)$ (Note that K_i now stands for sum of all elements in the class K_i). Then in terms of representation R the equation (4) becomes,

$$D(K_i)\ \ D(K_j) \ = \ \sum_{k=1}^{r} h_{ij,k}\ D(K_k) \tag{5}$$

Let the class K_i consists of elements $g_1, g_2 \ldots . g_{r_i}$

Then K_i of eq. (4) is,

$$K_i = g_1 + g_2 + \ldots\ldots + g_{r_i}$$

and $\quad D(K_i) = D(g_1) + D(g_2) + \ldots\ldots + D(K_i)$

are scalar matrices i.e. some scalar multiple of identity matrix $D(E) = I$ i.e. $D(K_i) = a_i I$ (a_i any number). So taking traces of the equation (5) one gets,

$$\text{tr.}\ (D(K_i)\ D(K_j)) \ = \sum_{k=1}^{r} h_{ij,k}\ \text{tr.}\ D(K_k)$$

or $\quad \text{tr.}\ (a_i\ a_j\ I) \ = \ \sum_{l=1}^{r} h_{ij,l}\ \text{tr}\ (a_l I)$

or $$a_i a_j d = \sum_{l=1}^{r} h_{ij,l}\, a_l d$$

But $$\text{tr } D(K_i) = \text{tr. } (D(g_1) + d(g_2) + \ldots . D(g_{r_i}))$$

$$= r_i k_i = d\, a_i$$

or $$a_i = \frac{r_i k_i}{d}$$

Therefore, $$r_i r_j k_i k_j = d \sum_{l=1}^{r} h_{ij,l}\, r_l k_l \qquad (6)$$

These equations determine all $k_1, k_2, \ldots k_r$. The above result is known as Burnside theorem. Note that the procedure of determining $h_{ij,k}$ has been sketched out in section 4.1.3. Some of the equations in eq. (6) are redundant and should be thrown out while determining k_i.

4.2.2 Some Relevant Results Related to Characters and Irreducible Representations

Following results which may be needed in calculating, irreducible representations, character etc. are quoted below (49-52).

1. A representation $R = \{ D(g_1), D(g_2), \ldots D(g_n) \}$ is irreducible if and only if,

$$\frac{1}{n} \sum_{i=1}^{n} \left| k\ (D(g_i)) \right|^2 = \frac{1}{n} \sum_{l=1}^{r} \left| r_l\ k_l \right|^2 = 1 \qquad (7)$$

where, $k\ (D(g_i))$ is the character of g_i in R,

r_l = the number of elements in the class K_l.

2. If a representation $R = \{D(g_i)\}$ is irreducible then

$$\sum_{i=1}^{n} D(g_i) \neq \sum_{l=1}^{r} r_l \, k_l = 0$$

3. If a representation $R = \{D(g_1), D(g_2) \ldots\ldots D(g_n)\}$ is irreducible of dimension d then there exist no non-zero, d-dimensional matrix D such that tr. $(D\, D(g_i)) = 0$ for all $i=1,2,\ldots\ldots n$.

4. If $R_1, R_2, \ldots\ldots R_r$ are irreducible representations of dimensions $d_1, d_2, \ldots\ldots d_r$ then $d_1^2 + d_2^2 + d_3^2 \ldots\ldots + d_r^2 = n \ldots$ .... (8)

where n is the order of the group.

5. The number of inequivalent irreducible representations of dimension 1 is given by (n/h_c), where h_c is the order of the commutator subgroup.

6. All irreducible reducible representations of an Abelian group are one dimensional and thus the number of irreducible representations is equal to the order of the group.

From now onward the p-th irreducible representation R_p will be written as $R_p = \{D_p(g_i)\}$ and corresponding characters will be written as k_{pi} .

7. If $R_p = \{D_p(g_i)\}$ and $R_q = \{D_q(g_i)\}$ are p-th and q-th irreducible representations then,

$$\sum_{l=1}^{n} D_{pij}(g_l)\, D_{qkm}(g_l) = \frac{g}{d_p}\, \delta_{pq}\, \delta_{ik}\, \delta_{jm} \qquad (9)$$

where δ_{ij} is Kronocker delta.

8. If a group G is a direct product of two groups H_1 and H_2 i.e. if $G = H_1 \times H_2$ and if P_1 and P_2 are the representations of the groups H_1 and H_2. Then the representation R of the group G is given by $R = P_1 \times P_2$ where $P_1 \times P_2$ is the direct products of the representation matrices. Then the character of the group G in representation R is given by,

$k^{(R)} = k^{(P_1 \times P_2)} = k^{(P_1)}\, k^{(P_2)}$ where k is the character of the group in the representation given by the superfix.

9. If a d-dimensional reducible representation $R' = \{D'(g_1)\ D'(g_2)\ \ldots\ D'(g_n)\}$ is reduced to $R = \{D(g_1),\ D(g_2)\ \ldots\ D(g_n)\}$ such that

$$R = c_1 R_1 \oplus c_2 R_2 \oplus c_3 R_3 + \ldots \oplus c_r R_r$$

where $R_1, R_2 \ldots R_r$ are irreducible representations of the group i.e. if $R = \{D(g_i)\}$

and $R_1 = \{D_1(g_i)\}$, $R_2 = \{D_2(g_i)\}$ .... $R_r = \{D_r(g_i)\}$

then $D(g_i) = c_1 D_1(g_i) + c_2 D_2(g_i) + \ldots c_r D_r(g_i)$

for $i = 1, 2, \ldots n$.

then

$$c_m = \frac{1}{n} \sum_{i=1}^{n} k(g_i)\, k_m\,(g_i) \qquad \text{-------------(10)}$$

Thus given any arbitrary representation $R = \{D(g_i)\}$ of a group $G = \{g_1, g_2, \ldots\, g_n\}$ one can immediately write it in reduced from by using equation (2) and (10). Thus an arbitrary set of representation matrices are block diagonalised in a minute. As remarked earlier, there is a direct connection between the block diagonalizations of representation matrices of the symmetry group of a structural system and the block diagonalization of stiffness matrix of the system. Since there exists a simple formula for the block diagonalisation of representation matrices of the symmetry group, one can speculate to have similar simple formula for the block diagonalisation of the stiffness matrix also. Before this work is pursued further it is preferable to illustrate as to what has been done so far by various examples.

4.2.3 <u>Illustrations</u>:

Consider the following groups one by one.

1 - The group C_{1h}:

The group multiplication table for this group is shown in Table 4.1(a).

There are two classes in the group namely (E)and (σ).

Therefore, there are two irreducible representations of the group. Also since thegroup is Abelian, it can have only one dimensional representations.

These are shown in Table 4.7.

The characters of the group is same as Table 4.7 because the representations are one-dimensional.

Consider now the reducible 3-dimensional representation R given by

$$D(E) = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix} \qquad D(\sigma) = \begin{bmatrix} 0 & & 1 \\ & -1 & \\ 1 & & 0 \end{bmatrix}$$

so, $k(E) = 3 \quad k(\sigma) = -1$

Then by eq. (2).

$$R = c_1 R_1 \oplus c_2 R_2$$

or $D(E) = c_1 D_1(E) \oplus c_2 D_2(E)$

$$D(\sigma) = c_1 D_1(\sigma) \oplus c_2 D_2(\sigma)$$

From eq. (10)

$$c_m = \frac{1}{n} \sum_{i=1}^{n} k(g_i)\, k_m(g_i)$$

$$c_m = \frac{1}{2} \left(k(E)\, k_m(E) + k(\sigma)\, k_m(\sigma)\right)$$

or $c_1 = \frac{1}{2} (3.1 + -1 \quad 1) = 1$

$$c_2 = \frac{1}{2} (3.1 + (-1).(-1)) = 2$$

TABLE 4.7

THE IRREDUCIBLE REPRESENTATIONS OF C_{1h}

	E	σ
R_1	1	1
R_2	1	-1

TABLE 4.8

THE IRREDUCIBLE REPRESENTATIONS OF THE GROUP C_{2v} (or the character Table)

	E	σ_1	σ_2	C_2
R_1	1	1	1	1
R_2	1	-1	1	-1
R_3	1	1	-1	-1
R_4	1	-1	-1	1

TABLE 4.9

THE IRREDUCIBLE REPRESENTATIONS OF THE GROUP C_{3v}.

	E	C_3	C_3^2	σ_v^a	σ_v^b	σ_v^c
R_1	1	1	1	1	1	1
R_2	1	1	1	-1	-1	-1
R_3	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$	$\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

TABLE 4.10

THE CHARACTER TABLE OF THE GROUP C_{3v}

	K_1 E (E)	K_2 (C_3, C_3^2)	K_3 $(\sigma_v^a, \sigma_v^b, \sigma_v^c)$
R_1	1	1	1
R_2	1	1	-1
R_3	2	1	0
R [illegible]	6	0	2

Thus the reduced representation of R is

$$D(E) = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix} \qquad D(\sigma) = \begin{bmatrix} 1 & & 0 \\ & -1 & \\ 0 & & -1 \end{bmatrix}$$

A similar argument holds for the group,

$C_2 = \{E, C_2\}$ also,

2. The group C_{2v}:

The group multiplication table to this group is shown in Table 4.2.

Since the group is Abelian as can be seen from the Table 4.2 ($\sigma_1 \sigma_2 = \sigma_2 \sigma_1$ etc. hold). Therefore every element of the group is a class by itself (see section 4.1.5) and so there are 4 classes. Therefore there will be 4-irreducible representations each one dimensional because if d_1, d_2, d_3, d_4 are the dimensions of irreducible representations R_1, R_2, R_3, R_4 then from eq. (8)

$d_1^2 + d_2^2 + d_3^2 + d_4^2 = 4$ and hence $d_1 = d_2 = d_3 = d_4 = 1$.

(This could have been argued from the properties of Abelian groups quoted in section 4.2.2.). The 4 inequivalent irreducible representations of the group are given in Table 4.8. Note that this table represents the characters also.

Consider now a 4n-dimensional representation R of the group,

$$D(E) = \begin{bmatrix} I & & & 0 \\ & I & & \\ & & I & \\ 0 & & & I \end{bmatrix} \qquad D(\sigma_1) = \begin{bmatrix} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{bmatrix}$$

$$D(\sigma_2) = \begin{bmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix} \qquad D(C_2) = \begin{bmatrix} 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \end{bmatrix}$$

where I is a n x n identity matrix.

Note that these representation matrices have been already encountered in III Chapter (section 3.2) and have been seen to satisfy the group postulates.

To get the reduced representation,

Let

$$R = c_1R_1 \oplus c_2 R_2 \oplus c_3 R_3 \oplus c_4 R_4$$

i.e $D(\sigma_1) = c_1D_1(\sigma_1) \oplus c_2D_2(\sigma_1) \oplus c_3D_3(\sigma_1) \oplus c_4D_4(\sigma_1)$

$D(\sigma_2) = c_1D_1(\sigma_2) \oplus c_2D_2(\sigma_2) \oplus c_3D_3(\sigma_2) \oplus c_4D_4(\sigma_2)$

$D(C_2) = c_1D_1(C_2) \oplus c_2D_2(C_2) \oplus c_3D_3(C_2) \oplus c_4D_4(C_2)$

where $D_k(g_i)$ are given in Table 4.8 $(i,k = 1,2,3,4)$.

The characters of the representation R are,

$$k(E) = 4n, \quad k(\sigma_1) = 0, \quad k(\sigma_2) = 0. \quad k(C_2) = 0$$

$$c_m = \frac{1}{4}\left(k(E)\, k_m(E) + k(\sigma_1)\, k_m(\sigma_1) + k(\sigma_2)\, k_m(\sigma_2) + k(C_2)\, k_m(C_2)\right)$$

$$= \frac{1}{4}\left(4n\, k_m(E)\right) \qquad \text{because } k(\sigma_1) = k(\sigma_2) = k(C_2) = 0$$

From table 4.8, $k_m(E) = 1$ for all $m = 1,2,3,4$.

Therefore $c_1 = c_2 = c_3 = c_4 = n$

So the reduced form of the matrices of R are

$$D(\sigma_1) = n\, D_1(\sigma_1) \oplus n\, D_2(\sigma_1) \oplus n\, D_3(\sigma_1) \oplus n\, D_4(\sigma_1)$$

$$D(\sigma_2) = n\, D_1(\sigma_2) \oplus n\, D_2(\sigma_2) \oplus n\, D_3(\sigma_2) \oplus n\, D_4(\sigma_2)$$

$$D(C_2) = nD_1(C_2) \oplus n\, D_2(C_2) \oplus n\, D_3(C_2) \oplus n\, D_4(C_2)$$

or

$$D(\sigma_1) = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & -I \end{bmatrix}, \qquad D(\sigma_2) = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & -I \end{bmatrix}$$

$$D(C_2) = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

3. The Group C_{3v}:

The group multiplication of this group is shown in Table 4.3. The order of this group is 6. There are 3-classes in this group, namely (E), $(C_3\ C_3^2)$ and $(\sigma_v^a, \sigma_v^b, \sigma_v^c)$ and therefore there will be 3-irreducible representations R_1, R_2 and R_3 of dimensions d_1, d_2 and d_3 respectively. From eq. (8) one must have,

$$d_1^2 + d_2^2 + d_3^2 = 6$$

The only possible integers satisfying the above eq. are 1, 1 and 2. So let $d_1 = d_2 = 1$ and $d_3 = 2$ i.e. there are two one-dimensional irreducible representations and there is one 2-dimensional irreducible representation. The one dimensional representations are:

R_1 : $D_1(E) = D_1(C_3) = D_1(C_3^2) = D_1(\sigma_v^a) = D_1(\sigma_v^b) = D_1(\sigma_v^c) = 1$

R_2 : $D_2(E) = D_2(C_3) = D_2(C_3^2) = 1$, $D_2(\sigma_v^a) = D_2(\sigma_v^b) = D_2(\sigma_v^c) = -1$

R_3 : $D_3(E) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $D_3(C_3) = ?$ $D(\sigma_v^a) = ?$ etc.

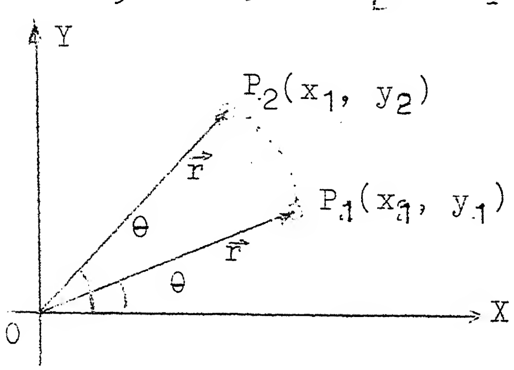


Fig. 4.1

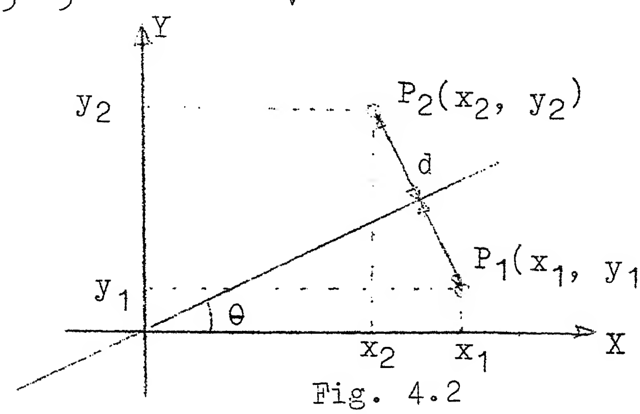


Fig. 4.2

In order to get $D_3(C_3)$, $D_3(\sigma_v^a)$ etc. the following arguments are necessary. Consider the Fig. 4.1. One has $x_1 = r \operatorname{Cos} \theta_1$, $y_1 = r \operatorname{Sin} \theta$, x_2 $r \operatorname{Cos} \theta_2$, $y_2 = r \operatorname{Sin} \theta_2$ and $\theta_2 = \theta + \theta_1$. Then from the identity,

$$\operatorname{Cos}(\theta + \theta_1) = \operatorname{Cos}\theta_1 \operatorname{Cos}\theta - \operatorname{Sin}\theta_1 \operatorname{Sin}\theta$$

$$\operatorname{Sin}(\theta + \theta_1) = \operatorname{Cos}\theta_1 \operatorname{Sin}\theta + \operatorname{Sin}\theta_1 \operatorname{Cos}\theta$$

One gets
$$\begin{aligned} x_2 &= x_1 \operatorname{Cos}\theta - y_1 \operatorname{Sin}\theta \\ y_2 &= x_1 \operatorname{Sin}\theta + y_1 \operatorname{Sin}\theta \end{aligned} \quad \text{or} \quad \begin{matrix} x_2 \\ y_2 \end{matrix} = \begin{matrix} \operatorname{Cos}\theta - \operatorname{Sin}\theta \; x_1 \\ \operatorname{Sin}\theta + \operatorname{Cos}\theta \; y_2 \end{matrix}$$

Thus the effect of rotations in two dimensional space is represented by the matrix,

$$D(C_\theta) = \begin{bmatrix} \operatorname{Cos}\theta & -\operatorname{Sin}\theta \\ \operatorname{Sin}\theta & \operatorname{Cos}\theta \end{bmatrix}$$ and hence by putting any value of $\theta = \dfrac{2\pi k}{n}$ one can get $D(C_n^k (k=1,2,\ldots.n-1,\ n=1,2,\ldots.)$

In the present case $\theta = 120°$ and $240°$ will solve the problem and the matrices,

$$D(C_3) = \begin{bmatrix} -(1/2) & -(\sqrt{3}/2) \\ (\sqrt{3}/2) & -(1/2) \end{bmatrix}, \quad D(C_3^2) = \begin{bmatrix} (1/2) & -(\sqrt{3}/2) \\ -(\sqrt{3}/2) & -(1/2) \end{bmatrix}$$

can be taken to be $D_3(C_3)$ and $D_3(C_3^2)$ respectively.

To get the matrices $D_3(\sigma_v^a)$ etc., consider the Fig. 4.2. One can see that,

$$x_2 = x_1 + 2d \operatorname{Sin} \theta$$

$$y_2 = y_1 - 2d \operatorname{Cos} \theta$$

Also $x_2^2 + y_2^2 = x_1^2 + y_1^2 + 4d^2 (\operatorname{Cos}^2\theta + \operatorname{Sin}^2\theta)$

$$- 4d(y_1 \operatorname{Sin}\theta - x_1 \operatorname{Cos} \theta) = x_1^2 + y_1^2$$

So $d = y_1 \operatorname{Sin}\theta - x_1 \operatorname{Cos} \theta$

or

$$x_2 = x_1 \operatorname{Cos} 2\theta + y_1 \operatorname{Sin} 2\theta$$
$$y_2 = x_1 \operatorname{Sin} 2\theta - y_1 \operatorname{Cos} 2\theta$$

or $\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} \operatorname{Cos}2\theta & \operatorname{Sin}2\theta \\ \operatorname{Sin}2\theta & -\operatorname{Cos}2\theta \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$

Thus reflection across a plane (in two dimension) inclined at $\theta°$ w.r.t. x-axis is represented by the matrix,

$$\begin{bmatrix} \operatorname{Cos}2\theta & \operatorname{Sin}2\theta \\ \operatorname{Sin}2\theta & -\operatorname{Cos}2\theta \end{bmatrix}$$

Consider now Fig. 2.2. For the reflection planes σ_v^a, σ_v^b and σ_v^c, $\theta = 90°$, $210°$ and $330°$ respectively and therefore the two dimensional representations of σ_v^a, σ_v^b, σ_v^c are following:

$$D_3(\sigma_v^a) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D_3(\sigma_v^b) = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \quad D(\sigma_v^c) = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

All of the matrices $D_3(C_3)$, $D_3(\sigma_v^a)$ etc. can not be further reduced by any similarity transformation whatsoever. (Note that a few of the 6-matrices could be reduced

through a similarity transformation by some matrix but not all of them at a time). Thus one can take these to represent the R_3. So the complete table of irreducible representation is as Table 4.9 reads.

Since R_1, R_2 are one dimensional, therefore the characters for R_1, R_2 are given by themselves, e.g. $k_1(C_3) = k_1(\sigma_v^a) = \ldots\ldots = 1$ and $k_2(C_3) = 1 \quad k_2(\sigma_v^a) = 1$. The characters of R_3 can be found either by use of eq. (6) or by taking traces of the matrices of R_3. One finds $tr.D_3(E)=2$ $tr.D_3(C_3) = tr.D_3(C_3^2) = 1$ and $tr.D_3(\sigma_v^a) = tr.D_3(\sigma_v^b)$ $= tr.D_3(\sigma_v^c) = 0$ (See table 4.9). One can see that characters are same for those elements which belong to a class e.g. $k_3\,(C_3) = k_3(C_3^2) = 1$ and $k_3(\sigma_v^a) = k_3(\sigma_v^c) = 0$ i.e. The characters of the classes $K_1 = (E)$ $K_2 = (C_3,\ C_3^2)$ and $K_3 = (\sigma_v^a,\ \sigma_v^b,\ \sigma_v^c)$ in R_3 are 2, 1 and 0 respectively. Using the double suffix notations for characters (where 1st suffix indicates the irreducible representation and second indicates the class) one has a matrix for the characters viz. $[k_{lm}]_{r \times r}$. This matrix is given the name "Character Table" of the group which for the group C_{3v} is a 3 x 3 matrix shown in Table 4.10.

In general the character table for a group with r-classes will have the following form:

$$\begin{array}{c|ccccc}
 & K_1 & K_2 & K_3 & \cdots & K_r \\
\hline
R_1 & k_{11} & k_{12} & k_{13} & \cdots & k_{1r} \\
R_2 & k_{21} & k_{22} & k_{23} & \cdots & k_{2r} \\
R_3 & k_{31} & k_{32} & k_{33} & \cdots & k_{3r} \\
- & - & - & - & - & - \\
R_r & k_{r1} & k_{r2} & k_{r3} & \cdots & k_{rr}
\end{array}$$

Comming back to C_{3v} consider now the 6-dimensional representation R given by

$$D(E)=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad D(C_3)=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$D(C_3^2) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and}$$

$$D(\sigma_v^a) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad D(\sigma_v^b) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\text{and } D(\sigma_v^c) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Characters of this representation are given by the last row of the Table 4.10, viz., 6, 0 and 2. From eq. (10) one can now immediately get the values of c_1, c_2 and c_3 for R , i.e.

$$c_m = \frac{1}{6} \sum_{i=1}^{6} k(g_i)\, k_m(g_i) = \frac{1}{6} \sum_{i=1}^{3} k(\ _i) k_{mi} \cdot r_i$$

$$c_m = \frac{1}{6} \left(k(K_1) k_{m1} + k(K_2)\, k_{m2}\ 2 + k(K_3) K_{m3}\ 3 \right)$$

$$= \frac{1}{6} \left(6K_{m1} + 6K_{m3} \right) \quad \text{because } k(K_1) = 6,\ k(K_3) = 2,$$

$$k(K_2) = 0.$$

So,

$$c_1 = k_{11} + k_{13} = 2$$

$$c_2 = k_{21} + k_{23} = 0$$

$$c_3 = k_{31} + k_{33} = 2$$

Thus $R = c_1R_1 \oplus c_2R_2 \oplus c_3R_3 = 2R_1 \oplus 2R_3$ and therefore the matrices of R reduce to the form;

$$D(g_i) = 2D_1(g_i) \oplus 2D_3(g_i) = D_1(g_i) \oplus D_1(g_i) \oplus D_3(g_i) \oplus D_3(g_i),$$

e.g.

$$D(C_3^2) = \left[\begin{array}{cc|cc|cc} 1 & 0 & & & & \\ 0 & 1 & & & & \\ \hline & & -(1/2) & (\sqrt{3}/2) & & \\ & & (\sqrt{3}/2) & -(1/2) & & \\ \hline & & & & -(1/2) & (\sqrt{3}/2) \\ & & & & (\sqrt{3}/2) & -(1/2) \end{array}\right]$$

and $$D(\sigma_v^c) = \left[\begin{array}{cc|cc|cc} 1 & 0 & & & & \\ 0 & 1 & & & & \\ \hline & & (1/2) & -(\sqrt{3}/2) & & \\ & & -(\sqrt{3}/2) & -(1/2) & & \\ \hline & & & & (1/2) & -(\sqrt{3}/2) \\ & & & & -(\sqrt{3}/2) & -(1/2) \end{array}\right]$$ etc.

All the matrices $D(g_i)$ are thus block diagonalised.

Similar arguments could be applied for any other representation of the group C_{3v}.

4. The Group C_{4v}:

The group multiplication table is 4.4. There are 5-classs, as shown in sub-section 4.1.3 in this group. Therefore there will be 5-irreducible representations R_1, R_2, R_3, R_4 and R_5 of dimensions d_1, d_2, d_3, d_4 and d_5 given by eq. (8), i.e.

$$d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2 = 8$$

The only possible values of d_1, ... d_5 are 1, 1, 1, 1 and 2. Thus four of the five irreducible representations are one-dimensional and one of them is 2-dimensional. These can be found by similar arguments as that for C_{3v} and are given in Table 4.11(a). The corresponding character table is given in Table 4.11(b).

Any other representation R can be writte as direct sum of the above irreducible representation.

5. The Cyclic Group C_m:

The group consists of E, C_m, C_m^2 ..... C_m^{m-1}. It can be seen that the group is Abelian and $C_m^m = E$. All the irreducible representations are one-dimensional and are in all, m.

If one choses the representations of C_m^k as θ_m^k the group multiplication table is satisfied. With this knowledge the table for m-inequivalent irreducible

TABLE 4.11 (a)

IRREDUCIBLE REPRESENTATION OF C_{4v}

	E	C_4	C_4^2	C_4^3	σ_1	σ_2	σ_3	σ_4
R_1	1	1	1	1	1	1	1	1
R_2	1	1	1	1	-1	-1	-1	-1
R_3	1	-1	1	-1	1	-1	1	-1
R_4	1	-1	1	-1	-1	1	-1	1
R_5	$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$	$\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$	$\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$	$\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$	$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$	$\begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}$	$\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$	$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$

TABLE 4.11 (b)

CHARACTER TABLE C_{4v}

K_1	K_2	K_3	K_4	K_5
1	1	1	1	1
1	1	1	-1	-1
1	1	-1	1	-1
1	1	-1	-1	1
2	-2	0	0	0

TABLE 4.12

THE IRREDUCIBLE REPRESENTATIONS OF C_m

	E	C_m	C_m^2	......	C_m^{m-1}
R_1	1	1	1	....	1
R_2	1	θ_1	θ_1^2	......	θ_1^{m-1}
R_3	1	θ_2	θ_2^2	....	θ_2^{m-1}
.	- - -	- - -	- - -	- - -	- - -
.					
R_m	1	θ_{m-1}	θ_{m-1}^2	.....	θ_{m-1}^{m-1}

representations can be written as Table 4.12.

Note that the Table 4.12 represents the character Table also because the group is Abelian.

Consider now nm-dimensional representation R of the group C_m given by either of the set of matrices

$T_1, T_2, T_3 \ldots T_m = I$ or $A_1, A_2, A_3 \ldots A_m = I$

of section 3.3. The characters of the representation given by $\{I, T_1, T_2, \ldots T_{m-1}\}$ are given by nm, 0, 0, ... 0 respectively. The reduced representation of $R = \{I, T_1 \ldots T_{m-1}\}$ is given by,

$$R = c_1R_1 \oplus c_2R_2 \oplus c_3R_3 \oplus \ldots \oplus c_mR_m$$

where

$$c_1 = \frac{1}{m} \sum_{i=1}^{m} k(C_m^i)\, k_1(C_m^i) = \frac{1}{m} k(E)\, k_1(E) = n$$

Thus $R = nR_1 \oplus nR_2 \oplus nR_3 \oplus + \ldots \oplus nR_m$ and hence the reduced forms of $T_1, T_2, T_3 \ldots T_{m-1}$ is atonce known. e.g. for T_1 and T_k the reduced forms are:

$$T_1 = \begin{bmatrix} I & & & & 0 \\ & I\theta_1 & & & \\ & & I\theta_2 & & \\ & & & \ddots & \\ 0 & & & & I\theta_{m-1} \end{bmatrix} \qquad T_r^k = \begin{bmatrix} I & & & & 0 \\ & I\theta_1^k & & & \\ & & I\theta_2^k & & \\ & & & \ddots & \\ 0 & & & & I\theta_{m-1}^k \end{bmatrix}$$

where I is n x n identity matrix.

Note that the matrices T_i are completely reduced i.e. they get diagonalised. This is the case with representation matrices of all Abelian groups. The irreducible representations and their character tables of various groups are given in the Appendix attached after the end of Chapter VI. The results of sub-section 4.2.2 will be very essential to determine the irreducible representations and character tables of any group.

Whatever has been done by now is giving no clue towards the applicability of various terms like groups, classes, representations, characters etc. The next section will indicate towards this aspect.

4.3 THE PHYSICAL MEANINGS OF REPRESENTATIONS

Consider a pphysical system with symmetry group $G = \{g_1, g_2, \ldots\ldots g_n\}$. Let there be N-degrees of freedom in the system. Let a basis $e = (e_1 \; e_2 \ldots e_N)$ is associated with these degrees of freedoms such that any displacement of the system as a whole is described by linear combinations of these basis vectors. These basis vectors can be thought of unit vectors at various nodes of a structural system, 6 at every node, three corresponding to displacements or forces and the rest three corresponding to

rotations or moments. Note that here two different kinds of space, the real three dimensional space in which displacement etc. at various nodes are visualised and the phase space in which the vectors are representing the displacement etc. for all the degrees of freedom, should be carefully distinguished.

Now consider the effects of symmetry elements $g_1, g_2, \ldots\ldots g_n$ on the basis vectors $e_1, e_2 \ldots\ldots e_N$. i.e. $g_i\ e_k$, $i=1, 2, \ldots\ldots n$, $k = 1, 2, \ldots. N$.

Since the basis $e_1, e_2, \ldots\ldots e_N$ is complete i.e. any vector can be written as the linear combination of these, therefore,

$$g_i\ e_k = \sum_{l=1}^{N} D_{lk}(g_i)\ e_l$$

Written in another way,

$$\begin{bmatrix} e_1 \\ e_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ e_N \end{bmatrix} = \begin{bmatrix} D_{11}(g_i) & D_{21}(g_i) & \ldots. & D_{N1}(g_i) \\ D_{12}(g_i) & D_{22}(g_i) & \ldots. & D_{N2}(g_i) \\ - - - & - - - & - - - & - - - \\ - - - & - - - & - - - & - - - \\ D_{1N}(g_i) & D_{2N}(g_i) & \ldots\ldots & D_{NN}(g_i) \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ e_N \end{bmatrix} \quad (i=1, 2, \ldots n$$

Consider, $g_i g_j e_k = g_i \sum_{l=1}^{N} e_l D_{lk}(g_j) = \sum_{l=1}^{N} \sum_{m=1}^{N} e_m D_{ml}(g_i) D_{lk}($

$$= \sum_{m=1}^{N} D_{mk}(g_l)\ e_m = g_l\ e_k$$

Therefore $g_i g_j = g_l \Rightarrow D(g_i)\, D(g_j) = D(g_l)$. Therefore the matrices $D(g_1)$, $D(g_2)$ .... $D(g_n)$ form a N-dimensional representation of the group G. Let this representation be called R. Thus the representation R defines a set transformation matrices which give the effect of symmetry operations on basis vectors e_1, e_2.... e_N. Also since the length must be preserved under symmetry operations, the matrices $D(g_i)$ must be or unitary or orthogonal as the case may be.

From the previous section it follows that the representation $R = \{D(g_i)\}$ can be reduced or can be written as direct sum of r irreducible representations $R_1 = \{D_1 (g_i)\}$, $R_2 = \{D_2(g_i)\}$, ... $R_r = \{D_r (g_i)\}$ of dimensions d_1, d_2.... d_r respectively where r is the number of classes in the group G.

or $$R = c_1 R_1 \oplus c_2 R_2 \oplus c_3 R_3 \oplus \ldots\ldots \oplus c_r R_r$$

or

$$R = \begin{bmatrix} R_1 & & & & & \\ & \ddots & & & & \\ & & R_1 & & & \\ & & & R_2 & & \\ & & & & \ddots & \\ & & & & & R_2 \\ & & & & & & R_r \\ & & & & & & & \ddots \\ & & & & & & & & R_r \end{bmatrix} \begin{array}{l} c_1 \text{ - times} \\ \\ c_2\text{-times} \\ \\ c_r\text{-times} \end{array}$$

where $R = D(g_i)$

The numbers $c_1, c_2 \ldots c_r$ are given by eq. (10), i.e.

$$c_m = \frac{1}{n} \sum_{i=1}^{n} k(g_i)\, k_m(g_i) = \frac{1}{n} \sum_{l=1}^{r} k(K_l)\, k_{ml}\, r_l$$

where K_l is the l-th class in the group, r_l is the number of element in K_l etc. The existence of reduced representation of R implies the existence of a basis in which the basis vectors are classified according to the transformation under symmetry operations, i.e. under all the symmetry operations of the group one set of vectors in the basis does not mix up with other sets of vectors in the basis and soon. This in turn needs a transformation of the first basis vectors to another one such that the resulting vectors in the basis have attained the above mentioned property. Let this new basis $\bar{e}$ is obtained from e by a transformation matrix S i.e.

$$\bar{e} = \begin{bmatrix} \bar{e}_1 \\ \bar{e}_2 \\ \vdots \\ \bar{e}_N \end{bmatrix} = \begin{bmatrix} S \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_N \end{bmatrix}$$

and $\bar{D}(g_i) = S\, D(g_i)\, S^{-1}$ are in reduced form.

The basis $\bar{e}$ is called the "symmetry addpted" basis. Since in the reduced form, $R_1, R_2 \ldots R_r$ occur $c_1, c_2, \ldots c_r$ times therefore the basis vectors can be classified in the following way: (by using a comprehensive notation).

$$\bar{e} = (\underbrace{\bar{e}_1^{(p,1)} \quad \bar{e}_2^{(p,1)} \ldots \bar{e}_{d_p}^{(p,1)}}_{\text{First occurrence of } R_p \text{ in } R}, \underbrace{\bar{e}_1^{(p,2)} \; \bar{e}_2^{(p,2)} \ldots \bar{e}_{d_p}^{(p}}_{\text{Second occurrence of } R_p \text{ in } R}$$

$$\underbrace{\bar{e}_1^{(1,1)} \; \bar{e}_2^{(1,1)} \ldots \bar{e}_{d_q}^{(q,1)}}_{\text{First occurrence of } R_q \text{ in } R}, \underbrace{\bar{e}_1^{(q,2)} \; \bar{e}_2^{(1,2)} \ldots \bar{e}_{d_q}^{(q,2)} \ldots}_{\text{Second occurrence of } R_q \text{ in } R})$$

Under the symmetry operations g_1, g_2, ... g_n the vectors $\bar{e}_1^{(p,1)}$ $\bar{e}_2^{(p,1)}$ .... $\bar{e}_{dp}^{(p,1)}$ are transformed by $D_p(g_1)$, $D_p(g_2)$ ...... $D_p(g_n)$ and are closed, similarly $\bar{e}_1^{(q,m)}$...... $\bar{e}_{d_q}^{(q,m)}$ are transformed by $D_q(g_1)$, $D_q(g_2)$ .... $D_q(g_n)$ and are closed for $l = 1, 2, \ldots c_p$, and $m = 1, 2, \ldots c_q$ for all p, and $q = 1, 2, \ldots r$. The basis vectors $\bar{e}_1^{(p,l)}$ $\bar{e}_2^{(p,l)}$ ..... $\bar{e}_{d_p}^{(p,l)}$ $(l=1,2, \ldots c_p)$ are said to be the basis of the irreducible representation $\{D_p(g_i)\}$ and are denoted by $e_i^{(p)}$ $(i=1,2, \ldots d_p)$ because under g_i for all l, they are transformed by $D_p(g_i)$. Note that these symmetry adapted basis vectors are linear combinations of the original basis vectors whose co-efficients are the elements of unknown matrix S which transorms R to reduced form. The basic aim is to find out this matrix or "the symmetry adapted basis". Group theory provides this symmetry adapted basis very

easily. In $\bar{e}_i^{(p,l)}$ the pair (p, i) is called the "symmetry species" of the basis and l distinguishes different vectors of the same species.

4.3.1 Generation of the Symmetry Adapted Basis:

Consider an arbitrary vector V in the phase space. This can be written as

$$V = \sum_{i=1}^{N} V_i \, \bar{e}_i = \sum_{q=1}^{r} \sum_{k=1}^{d_q} \sum_{l=1}^{c_q} v_k^{(q,l)} \, \bar{e}_k^{(q,l)}$$

$$= \sum_q \sum_k V_k^{(q)} \quad \text{where} \quad V_k^{(q)} = \sum_{l=1}^{c_q} v_k^{(q,l)} \, \bar{e}_k^{(q,l)}$$

and so $V_k^{(q)}$ lies in the plane spaned by $\bar{e}^{(q,1)}$ $\bar{e}_k^{(q,2)}$ .... $\bar{e}_k^{(1,c_q)}$ which have got exactly similar transformations under g_i. There are $d_1 + d_2 + \ldots + d_r$ such different planes or subspaces (for every q, d_q planes or sub-spaces). The aim is to extract out one plane from all these $d_1 + d_2 + \ldots + d_r$ planes. Consider the following,

$$V' = \sum_{i=1}^{n} D_{pjk}(g_i) \, g_i V$$

$$= \sum_{i=1}^{n} D_{pjk}(g_i) \, g_i \sum_{q=1}^{r} \sum_{s=1}^{d_q} \sum_{m=1}^{c_q} v_s^{(q,m)} \, \bar{e}^{(q,m)}$$

By definition, $g_i \, \bar{e}_s^{(q,m)} = \sum_{l=1}^{n} \bar{e}_l^{(q,m)} \, D_{qls}(g_i)$

Therefore, $$V' = \sum_{i=1}^{n} \sum_{q=1}^{r} \sum_{m=1}^{c_q} \sum_{s=1}^{d_q} \sum_{l=1}^{n} \Big(D_{pjk}(g_i) D_{qls}(g_i) \cdot v_s^{(q,m)} \, \bar{e}_l^{(l,m)} \Big)$$

From eq. (9) (Section 4.2.2) one has ,

$$\sum_{i=1}^{n} D_{pjk}(g_i) \, D_{qms}(g_i) = \frac{n}{d_p} \cdot \delta_{pq} \, \delta_{jm} \, \delta_{ks},$$

Therefoe

$$V' = \sum_{q=1}^{r} \sum_{s=1}^{d_q} \sum_{m=1}^{c_q} \sum_{l=1}^{c_q} \frac{n}{d_p} \, \delta_{pq} \, \delta_{jl} \, \delta_{ks} \, v_s^{(q,m)} \, \bar{e}_l^{(q,m)}$$

$$= \frac{n}{d_p} \sum_{m=1}^{c_p} v_k^{(p,m)} \, \bar{e}_j^{(p,m)}$$

or $V' = \sum_{i=1}^{n} D_{pjk}(g_i) g_i \, V = \frac{n}{d_p} \sum_{m=1}^{c_p} v_k^{(p,m)} \, \bar{e}_j^{(p,m)}$

But the $\bar{e}_j^{(p,1)}$ $\bar{e}_j^{(,2)}$, ... $\bar{e}_j^{(p,c_p)}$ for (fixed p, j) have got same transformations under g_i and are one of the basis vectors for D_p. The r sum will also have same transformation property and therefore $\frac{n}{d_p} \sum_{m=1}^{c_p} v_k^{(p,m)} \bar{e}_j^{(p,m)}$ are basis vectors $e_1^{(p)}$, $e_2^{(p)}$, ... $e_{d_p}^{(p)}$ for irreducible representation $\{D_p(g_i)\}$.

Thus $\sum_{i=1}^{n} D_{pjk}(g_i)g_i \, V \sim e_j^{(p)}$ (for fixed k, $j=1,2,\ldots d_p$)

where $\sim$ means the same transformation property. The basis for D_p are therefore generated by $\sum_{i=1}^{n} D_{pjk}(g_i)g_i$ if it is applied on any arbitrary vector. This operator thus projects any vector on the basis vectors of the irreducible representations and is therefore called PROJECTION OPERATOR denoted by $P_{jk}^{(p)}$

$$P_{jk}^{(p)} = \sum_{i=1}^{n} D_{pjk}(g_i)\, g_i \quad \text{and} \quad P_{jj}^{(p)} = (\sum_{i=1}^{n} D_{pjj}(g_i)g_i \text{ ---(11)}$$

$$P_{jk}^{(p)}V = e_j^{(p)} \quad \text{and} \quad P_{jj}^{(p)} \; V = e_j^{(p)} \text{ ----------- (12)}$$

The eqs. (11) and (12) are probably the most important eqs. of this thesis. They generate the basis in which the representation R gets reduced and as will be seen, this basis will be one in which the stiffness matrix of the structural system gets block diagonalised. Not only this, the basis will provide the normal modes of the structural system.

The special case of projection operator, $P_{jj}^{(p)}$ known as IDEMPOTENT OPERATOR is equally important because the knowledge of this is sufficient to generate the complete basis and those $P_{jk}^{(p)}$ for $j \neq k$ are redundant so far as the generation of basis is concerned. If one takes the trace of the equation (11) one gets another operator.

known as "Character projection operator". This is also of importance and is

$$P^{(p)} = \sum_{i=1}^{n} k_p(g_i)g_i = \sum_{l=1}^{n} k_{pl} K_l \qquad (13)$$

where K_l is the class sum of class K_l.

Now the basis in which the representation R reduces is at hand and one has sufficient tool to simplify the symmetrical structural problems. The next chapter will be devoted to this aspect. Before this is done, an illustration of the above procedure is given for the structural systems with the symmetry group $C_{1h} = \{E, \sigma\}$ The irreducible representation of this group are given in Table 4.7. From eq. (11) $P_{ij}^{(p)} = \sum_{l=1}^{n} D_{pij}(g_l)\, g_l$,

for this case is,

$$P^{(1)} = 1 \cdot E + 1.\sigma = E + \sigma \quad (D_p \text{ are } 1 \times 1 \text{ matrices})$$

$$P^{(2)} = 1 \cdot E - 1.\sigma = E - \sigma$$

Let the original basis of the system be $(e_1, e_2 \ldots e_N)$. Then the symmetry adapted basis is given by N linearly independent non-zero vectors generated by,

$$P^{(1)}.e_1,\ P^{(1)}.e_1 \ldots .\ P^{(1)}.e_N \quad \text{and} \quad P^{(2)}.e_1,\ P^{(2)}.e_2,\ \ldots .\ P^{(2)}.e_N .$$

Since $Ee_i = e_i$ and let the original basis is such that $\sigma e_1 = -e_N$, $\sigma e_2 = -e_{N-1} \ldots\ldots \sigma e_{N/2} = -e_{\frac{N}{2}+1}$

or $\sigma e_{\frac{N+1}{2}} = -e_{\frac{N+1}{2}}$ as N is even or odd repectively.

Then for N = even, $P^{(1)} e_i = (E + \sigma)\, e_i = e_i - e_{N-i+1}$

$(i=1,2,, \ldots N)$

$P^{(2)} e_i = (E - \sigma)\, e_i = e_i + e_{N-i+1} \qquad i=1,2, \ldots N.$

Therefore the symmetry adopted basis for N = even is:

$$\left(\frac{1}{\sqrt{2}}(e_1 \pm e_N),\ \frac{1}{\sqrt{2}}(e_2 \pm e_{N-1}) \ldots \frac{1}{\sqrt{2}}(e_{N/2} \pm e\ \tfrac{N}{2} + 1)\right)$$

For N = odd the symmetry adapted basis is,

$$\left(\frac{1}{\sqrt{2}}(e_1 \pm e_N),\ \frac{1}{\sqrt{2}}(e_2 \pm e_{N-1}) \ldots \frac{1}{\sqrt{2}}(e_{\frac{N-1}{2}} \pm e_{\frac{N+3}{2}}),\ e_{\frac{N+1}{2}}\right)$$

It is seen here that even if the node comes on the reflection plane the arguments of symmetry remain fully applicable (N=odd here). Note that the initial choice of basis vectors may be arbitrary and still the symmetry adapted basis can be obtained. To this end 2 lemmas due to Schur must be noted here:

(i) Given any two irreducible representations $R_1 = \{D_1(g_i)\}$ and $R_2 = \{D_2(g_i)\}$, then for some matrix S, the relation $D_1(g_i)S = S\, D_2(g_i)$ (for all $i=1,2,\ldots n$) holds only if

either S is a null matrix or it is a square non-singular matrix.

(ii) The only non-trivial matrix which commutes will all the matrices of an irreducible representation, is a multiple of identity matrix.

The later lemma follows from the former. Both of them will be useful in the next chapter.

CHAPTER 5

APPLICATION OF GROUP THEORY TO SYMMETRICAL STRUCTURAL SYSTEMS

5.1 GENERAL CONSIDERATIONS:

Consider a structural system with a symmetry group G. Let there be N-degrees of freedom. Let some arbitrary co-ordinate systems are chosen whose basis vectors are $e_1, e_2, \ldots\ldots e_N$. Out of these N-tple vectors 6,5,4,3,2 or 1 belong to every node as the case may be. Let the structural system deform and the nodal displacement vector X is described by a set of generalised co-ordinates $q_1, q_2 \ldots q_N$ referred to the above basis. The deformation potential energy $U = U(q_1, q_2 \ldots\ldots q_N) = U(X)$ may be any positive function of X such that $U(X) = U(-X)$. Now since the system has symmetry group $G = \{g_1, g_2 \ldots\ldots g_n\}$, therefore the behaviour of system under $g_1, g_2, \ldots g_n$ should remain invariant because these symmetry operations by $g_1, g_2 \ldots g_n$ are equivalent to rigid body rotations or displacements of the system which does not produce any deformation energy. Thus the deformation energy $U(X)$ must remain invariant under all the symmetry operations of the symmetry group G, which in turn means,

$$U(g_1X) = U(X), \quad U(g_2X) = U(X) \quad \ldots\ldots U(g_nX) = U(X)$$

But $X = q_1e_1 + q_2e_2 + \ldots\ldots + q_N e_n$ and as has been seen in

Chapter 4 that $g_i e_k = \sum_{l=1}^{N} D_{lk}(g_i)\, e_l$.

Therefore $g_i X = D(g_i)\, X$ and so the invariance under $g_1, g_2 \ldots\ldots g_n$ means invariance under $D(g_1), D(g_2) \ldots\ldots D(g_n)$ where $\{D(g_i)\}$ is the N-dimensional representation of the group G. Let this representation be called R. Thus for a structural system with symmetry group G whose N-dimensional representation is R, $\overline{U}(RX) = U(X)$ i.e.

$$U(D(g_i)X) = U(X) \quad (i=1,2,\ldots.n) \qquad (1)$$

Eqs. (1) is a set of n-equations out of which only few may be independent and the rest may follow from those few. However it should not be taken that the dependent set are useless and should be thrown away. The moment any member of $\{D(g_i)\}$ or G is thrown away from considerations one will lose one's track and the whole effort will either be a waste or will yield less information than could have been obtained from the use of thecomplete group. At this moment one can note that if instead of using the complete group one uses subgroups of it, one will get different kinds of simplifications depending upon the nature of the sub-groups used. e.g. if G has sub-groups $H_1, H_2, H_3 \ldots\ldots H_s$ of order $s_1, s_2, s_3 \ldots\ldots$ respectively and if their N-dimensional representations are $P_1 = \{D(h_i^{(1)})\}$, $P_2 = \{D(h_i^{(2)})\} \ldots\ldots P_s = \{D(h_i^{(s)})\}$, then the invariance of $U(X)$ under $H_1, H_2 \ldots\ldots H_s$ means.

$$U(D(h_i^{(1)})X) = U(X) \; ; \quad i=1, 2, \ldots s_1$$
$$U(D(h_i^{(2)})X) = U(X); \quad i=1,2,\ldots s_2 \qquad (2)$$
$$- \; - \; - \; - \; - \; - \; - \; - \; - \; -$$
$$U(D(h_i^{(s)})X) = U(X); \quad i=1,2, \ldots s_s$$

The resulting simplifications by use of different subgroups H_1, H_2 .... H_s may be altogether different because they may have altogether different nature and hence different irreducible representations etc. Not only this, use of a subgroup of a symmetry group may yield more ease in calculations than that of the full group. The reason is that finding of the symmetry ad pted basis may be easier for a subgroup than that of the full group but ofcourse the information obtained will be less. As an illustration to this consider the system shown in Fig. 3.10(a). This has symmetry group C_{3v} and there are 6-degrees of freedoms. Now if one wants to use C_{3v} one has to put some effort to find the symmetry adapted basis, while if one uses only the subgroup C_3, one can visualise that the chosen basis in Chapter III itself is symmetry adapted and hence much of effort is saved this side but as will be seen latter that the other calculations are easier if one uss the C_{3v}.

The equation (1) is sufficiently general and is true for linear and nonlinear problems both. The nonlinear problems are how far reduced by use of group theory is to

$R = c_1R_1 \oplus c_2R_2 \oplus \ldots\ldots \oplus c_rR_r$ where $R_1 = \{D_1(g_i)\}$, $R_2 = \{D_2(g_i)\}$ ..... $R_r = \{D_r(g_i)\}$ are the irreducible representations of the group. To make the matter more explicit consider the following cases.

(1) The group has two class and hence two irreducible representations R_1 and R_2 of dimensions d_1 and d_2.

(i) If R_1 and R_2 occur only once in R i.e.

$$R = R_1 \oplus R_2 = \begin{bmatrix} R_1 & \\ & R_2 \end{bmatrix}$$

Then if $K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$ is partioned as R,

KR = RK means $\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} R_1 & \\ & R_2 \end{bmatrix} = \begin{bmatrix} R_1 & \\ & R_2 \end{bmatrix} \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$

or $\begin{bmatrix} K_{11}R_1 & K_{12}R_2 \\ K_{21}R_1 & K_{22}R_2 \end{bmatrix} = \begin{bmatrix} R_1K_{11} & R_1K_{12} \\ K_2K_{21} & R_2K_{22} \end{bmatrix}$

or $K_{11}R_1 = R_1K_{11}$, $K_{22}R_2 = R_2K_{22}$

$K_{12}R_2 = R_1K_{12}$ $\qquad K_{21}R_1 = R_2K_{21}$

for $R_1 = \{D_1(g_i) \quad i=1, 2, \ldots. n\}$ and $R_2 = \{D_2(g_i) \quad i=1,2, \ldots. n)\}$

From Schur lemmas (please see at the end of IV Chapter) it follows that,

$K_{11} = k_1 I_1$, $K_{22} = k_2 I_2$, $K_{12}' = K_{21} = 0$ where I_1 and I_2 are $d_1 x d_1$ and $d_2 x d_2$ identity matrices. Thus

$$K = \left[\begin{array}{cc} k_1 I_1 & 0 \\ \hline 0 & k_2 I_2 \end{array}\right] = \begin{bmatrix} k_1 & & & & & & \\ & k_1 & & & & & \\ & & \ddots & & & & \\ & & & k_1 & & & \\ & & & & k_2 & & \\ & & & & & \ddots & \\ & & & & & & k_2 \end{bmatrix}$$ i.e.

(ii) Now let R_1 is occurring twice and R_2 once i.e.

$$R = 2R_1 \oplus R_2 = \begin{bmatrix} R_1 & & \\ & R_1 & \\ & & R_2 \end{bmatrix}.$$ The equation (4),

when K is also partitioned in similar way is,

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \begin{bmatrix} R_1 & & \\ & R_1 & \\ & & R_2 \end{bmatrix}$$

$$= \begin{bmatrix} R_1 & & \\ & R_1 & \\ & & R_2 \end{bmatrix} \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}$$

or $$\begin{bmatrix} K_{11} & R_1 & K_{12} & R_1 & K_{13} & R_2 \\ K_{21} & R_1 & K_{22} & R_1 & K_{23} & R_2 \\ K_{31} & R_1 & K_{32} & R_1 & K_{33} & R_2 \end{bmatrix} = \begin{bmatrix} R_1 & K_{11} & R_1 & K_{12} & R_1 & K_{13} \\ R_1 & K_{21} & R_1 & K_{22} & R_1 & K_{23} \\ R_2 & K_{31} & R_2 & K_{32} & R_2 & K_{33} \end{bmatrix}$$

From Schur lemmas it follows that,

$K_{11} = k_1 I_1$, $K_{22} = k_2 I_1$, $K_{33} = k_4 I_2$, $K_{12} = k_3 I_1 = K_{21}$

$K_{13} = K_{31}$; $K_{23} = K_{32} = 0$ where I_1 and I_2 and $d_1 \times d_1$ and $d_2 \times d_2$ identity matrices.

Then
$$K = \left[\begin{array}{cc|c} k_1 I_1 & k_3 I_1 & \\ k_3 I_1 & k_2 I_1 & \\ \hline & & k_4 I_2 \end{array}\right]$$
Let $d_1 = d_2 = 2$ then,

$$K = \left[\begin{array}{cccc|cc} k_1 & 0 & k_3 & 0 & & \\ 0 & k_1 & 0 & k_3 & & \\ k_3 & 0 & k_2 & 0 & & \\ 0 & k_3 & 0 & k_2 & & \\ \hline & & & & k_4 & 0 \\ & & & & 0 & k_4 \end{array}\right] = \left[\begin{array}{cccccc} k_1 & k_3 & & & & \\ k_3 & k_2 & & & & \\ & & k_1 & k_3 & & \\ & & k_3 & k_2 & & \\ & & & & k_4 & 0 \\ & & & & 0 & k_4 \end{array}\right]$$

which is obtained by re-arranging the rows and columns according to symmetry species of R_1 i.e. the 3rd row is brought just below the Ist row and the third column is brought just right to the Ist column etc. This type of

interchanges of rows and columns is equivalent to the arrangements of corresponding basis vectors which bring the stiffness matrix into block forms as can be seen from following little more involved example:

(2) Let R_1, R_2 and R_3 be d_1, d_2 and d_3 dimensional irreducible representations and are contained in R 2-times each i.e.

$$R = \begin{bmatrix} R_1 &&&&& \\ & R_1 &&&& \\ && R_2 &&& \\ &&& R_2 && \\ &&&& R_3 & \\ &&&&& R_3 \end{bmatrix} \begin{array}{l} \text{in basis } (\bar{e}_1^{(1,1)}\ \bar{e}_2^{(1,1)} \ldots\ \bar{e}_{d_1}^{(1,1)}) \\ \text{in basis } (\bar{e}_1^{(1,2)}\ \bar{e}_2^{(1,2)} \ldots\ \bar{e}_{d_1}^{(1,2)}) \\ \text{in basis } (\bar{e}_1^{(2,1)}\ \bar{e}_2^{(2,1)} \ldots\ \bar{e}_{d_1}^{(1,2)}) \\ \text{in basis } (\bar{e}_1^{(2,2)}\ \bar{e}_2^{(2,2)} \ldots\ \bar{e}_{d_2}^{(2,2)}) \\ \text{in basis } (\bar{e}_1^{(3,1)}\ \bar{e}_2^{(3,1)} \ldots\ \bar{e}_{d_3}^{(3,1)}) \\ \text{in basis } (\bar{e}_1^{(3,2)}\ \bar{e}_2^{(3,2)} \ldots\ \bar{e}_{d_3}^{(3,2)}) \end{array}$$

then in this basis KR = RK m

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} \end{bmatrix} \begin{bmatrix} R_1 &&&&& \\ & R_1 &&&& \\ && R_2 &&& \\ &&& R_2 && \\ &&&& R_3 & \\ &&&&& R_3 \end{bmatrix}$$

$$
= \begin{bmatrix} R_1 & & & & & \\ & R_1 & & & & \\ & & R_2 & & & \\ & & & R_2 & & \\ & & & & R_3 & \\ & & & & & R_3 \end{bmatrix} \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} \end{bmatrix}
$$

or

$$
\left[\begin{array}{cccccc} K_{11}R_1 & K_{12}R_1 & K_{13}R_2 & K_{14}R_2 & K_{15}R_3 & K_{16}R_3 \\ K_{21}R_2 & K_{22}R_1 & K_{23}R_2 & K_{24}R_2 & K_{25}R_3 & K_{26}R_3 \\ K_{31}R_1 & K_{32}R_1 & K_{33}R_2 & K_{34}R_2 & K_{35}R_3 & K_{36}R_3 \\ K_{41}R_1 & K_{42}R_1 & K_{43}R_2 & K_{44}R_2 & K_{45}R_3 & K_{46}R_3 \\ K_{51}R_1 & K_{52}R_1 & K_{53}R_2 & K_{54}R_2 & K_{55}R_3 & K_{56}R_3 \\ K_{61}R_1 & K_{62}R_1 & K_{63}R_2 & K_{64}R_2 & K_{65}R_3 & K_{66}R_3 \end{array}\right]
\begin{array}{l} \text{in basin } (\bar{e}^{(1,1)} \ldots e_{d_1}^{(1,1)}) \\ \text{in basin } (\bar{e}_1^{(1,2)} \ldots \bar{e}_{d_1}^{(1,2)}) \\ \text{in basin } (\bar{e}_1^{(2,1)} \ldots \bar{e}_{d_2}^{(2,1)}) \\ \text{in basin } (\bar{e}_1^{(2,2)} \ldots \bar{e}_{d_2}^{(2,2)}) \\ \text{in basin } (\bar{e}_1^{(3,1)} \ldots \bar{e}_{d_3}^{(3,1)}) \\ \text{in basin } (\bar{e}_1^{(3,2)} \ldots \bar{e}_{d_3}^{(3,2)}) \end{array}
$$

$$
= \begin{bmatrix} R_1K_{11} & R_1K_{12} & R_1K_{13} & R_1K_{14} & R_1K_{15} & R_1K_{16} \\ R_1K_{21} & R_1K_{22} & R_1K_{23} & R_1K_{24} & R_1K_{25} & R_1K_{26} \\ R_2K_{31} & R_2K_{32} & R_2K_{33} & R_2K_{34} & R_2K_{35} & R_2K_{36} \\ R_2K_{41} & R_2K_{42} & R_2K_{43} & R_2K_{44} & R_2K_{45} & R_2K_{46} \\ R_3K_{51} & R_3K_{52} & R_3K_{53} & R_3K_{54} & R_3K_{55} & R_3K_{56} \\ R_3K_{61} & R_3K_{62} & R_3K_{63} & R_3K_{64} & R_3K_{65} & R_3K_{66} \end{bmatrix} \quad \text{in the basis as above.}
$$

By applying Schur lemmas one gets,

$K_{11} = k_{11}I_1$, $K_{22} = k_{22}I_1$, $K_{33} = k_{33}I_2$, $K_{44} = k_{44}I_2$,

$K_{55} = k_{55}I_3$, $K_{66} = k_{66}I_3$, $K_{12} = k_{12}I_1$, $K_{13} = K_{14} = K_{15} = K_{16}$

$= K_{31} = K_{41} = K_{51} = K_{61} = K_{23} = K_{24} = K_{25} = K_{26} = K_{32} = K_{42} = K_{52}$

$= K_{62} = K_{35} = K_{36} = K_{53} = K_{63} = K_{45} = K_{46} = K_{54} = K_{64} = 0.$

$K_{34} = K_{43} = k_{34} I_2$, $K_{56} = K_{65} = k_{56}I_3$ and therefore,

$$K = \begin{bmatrix} k_{11}I_1 & k_{12}I_1 & & & & \\ k_{12}I_1 & k_{22}I_1 & & & & \\ & & k_{33}I_2 & k_{34}I_2 & & \\ & & k_{34}I_2 & k_{44}I_2 & & \\ & & & & k_{55}I_3 & k_{56}I_3 \\ & & & & k_{56}I_3 & k_{66}I_3 \end{bmatrix}$$

where the above blocks are referred to the basis

$(\bar{e}^{(1,1)}, \bar{e}_2^{(1,1)} \ldots\ldots \bar{e}_{d_1}^{(1,1)}), (\bar{e}_2^{(1,2)}\ \bar{e}_2^{(1,2)} \ldots \bar{e}_{d_1}^{(1,2)})$

and

$(\bar{e}_1^{(2,1)}\ \bar{e}_2^{(2,1)} \ldots \bar{e}_{d_2}^{(2,1)}), (\bar{e}_1^{(2,2)} \bar{e}_2^{(2,2)} \ldots \bar{e}_{d_2}^{(2,2)})$ and

$(\bar{e}_1^{(3,1)}\ \bar{e}^{(3,1)} \ldots \bar{e}_{d_3}^{(3.1)}), (\bar{e}_1^{(3,2)}, \bar{e}_2^{(3,2)} \ldots\ldots \bar{e}_{d_3}^{(3,2)})$

respectively.

where I_1, I_2 and I_3 are $d_1 \times d_1$, $d_2 \times d_2$ and $d_3 \times d_3$ identity matrices. This form of stiffness matrix is ofcourse very simple but is in big blocks of dimensions $2d_1 \times 2d_1$, $2d_2 \times 2d_2$ and $2d_3 \times 2d_3$. Now let the vectors in the basis of R_1, R_2 and R_3 are arranged according to their respective symmetry species i.e. the set of vectors in the complete basis are arranged as follows:

$$\bar{e} = (\underbrace{\bar{e}_1^{(1,1)}\ \bar{e}_1^{(1,2)}}_{(1,1)},\ \underbrace{\bar{e}_2^{(1,1)}\ \bar{e}_2^{(1,2)}}_{(1,2)},\ \ldots\ldots\ \underbrace{\bar{e}_{d_1}^{(1,1)}\ \bar{e}_{d_1}^{(1,2)}}_{(1,d_1)};$$

metry species: (1,1) (1,2) $(1,d_1)$

$$\underbrace{\bar{e}_1^{(2,1)}\ \bar{e}_1^{(2,2)}}_{(2,1)},\ \underbrace{\bar{e}_2^{(2,1)}\ d\bar{e}_2^{(2,2)}}_{(2,2)}\ \ldots\ldots\ \underbrace{\bar{e}_{d_2}^{(2,1)}\ \bar{e}_{d_2}^{(2,2)}}_{(2,d_2)};$$

$$\underbrace{\bar{e}_1^{(3,1)}\ \bar{e}_1^{(3,2)}}_{(3,1)},\ \underbrace{\bar{e}_2^{(3,1)}\ \bar{e}_2^{(3,2)}}_{(3,2)}\ \ldots\ldots\ \underbrace{\bar{e}_{d_3}^{(3,1)}\ \bar{e}_{d_3}^{(3,1)}}_{(3,d_3)}$$

This arrangement means a corresponding interchanges in rows and columns. Here the 2nd and d_1 + 1th rows and column interchange, 4th and d_1+3rd rows and column interchange, 6th rwo and d_1 + 5th rows and column interchange etc. Under this the stiffness matrix will assume the form (for d_1=2, d_2=1, and d_3 = 2).

$$K = \left[\begin{array}{cccccccccc}
k_{11} & k_{12} & & & & & & & & \\
k_{21} & k_{22} & & & & & & & & \\
 & & k_{11} & k_{12} & & & & & & \\
 & & k_{21} & k_{22} & & & & & & \\
 & & & & k_{33} & k_{34} & & & & \\
 & & & & k_{34} & k_{44} & & & & \\
 & & & & & & k_{55} & k_{56} & & \\
 & & & & & & k_{56} & k_{66} & & \\
 & & & & & & & & k_{55} & k_{56} \\
 & & & & & & & & k_{56} & k_{66}
\end{array}\right]
\begin{array}{l}
R_1, \text{Ist vector,} \\
\quad \text{species } (1,1) \\
R_1, \text{IInd vector} \\
\quad \text{species } (1,2) \\
R_2, \text{Ist vector} \\
\quad \text{species } (2,1) \\
R_3, \text{Ist vector} \\
\quad \text{species } (3,1) \\
R_3, \text{ IInd Vector} \\
\quad \text{specie } (3.2)
\end{array}$$

From this and the earlier examples it is clear that the stiffness matrix **breaks** to blocks whose sizes are equal to the number of occurrence of irreducible representations. Thus in the above R_1 and R_3 has diemsnion 2 therefore there are 2-identical blocks corresponding to R_1 and R_3 while there is only one block corresponding to R_2 because its dimension is 1.

In general case where there are r-irreducible representations $R_1, R_2, \ldots . R_r$ occurring $c_1, c_2, \ldots . c_r$ times in R respectively i.e. $R = c_1R_1 \oplus c_2 R_2 \oplus c_3R_3 \oplus \ldots . \oplus c_rR_r$, let the basis in which R reduces be

$$\bar{e} = (\ldots\ldots \bar{e}_1^{(p,1)}, \bar{e}_2^{(p,1)} \ldots\ldots \bar{e}_{d_p}^{(p,1)}; \bar{e}_1^{(p,2)}\, \bar{e}_2^{(p,2)}$$

$$\ldots\ldots\ldots \bar{e}_{d_p}^{(p,2)} \ldots\ldots\ldots)$$

and let it be arranged according to the symmetry species (p,i). This arranged basis is,

$$\bar{e} = (\ldots\ \bar{e}_1^{(p,1)}\ \bar{e}_1^{(p,2)}\ \ldots\ \bar{e}_1^{(p,c_p)}\ ;\ \bar{e}_2^{(p,1)}, \bar{e}_2^{(p,2)} \ldots \bar{e}_2^{(p,c_p)})$$

Ist p-th block IInd p-th block ....

(p=1,2,....r)

Then corresponding to R_p in the stiffness matrix there will be d_p indentical blocks of $c_p \times c_p$. Thus the whole stiffness matrix breaks to:

d_1	identical	blocks	of $c_1 \times c_1$	corresponding	to	R_1
d_2	"	"	" $c_2 \times c_2$	"	"	R_2
d_3	"	"	" $c_3 \times c_3$	"	"	R_3
- - -	- - -	- - -	- - -	- - -	- - -	- - -
d_r	"	"	" $c_r \times c_r$	"	"	R_r.

Therefore the inversion etc. of the N x N stiffness matrix boils down to inversions of some or all of the following r differnt matrices:

One matrix of order $c_1 \times c_1$

One matrix of order $c_2 \times c_2$

. . . .

One matrix of order $c_r \times c_r$

where r is the number of classes in the group.

Thus even without knowing any thing about the stiffness matrix, one can know the resulting simplifications if one uses

the symmetry adapt basis arranged according to their symmetry species. The only knowledge needed at this stage is, the knowledge about the symmetry group G(i.e. the character table and the characters of the N-dimensional representation). The d_1, d_2, ... d_r are known from the knowledge of dimensionality of the irreducible representations which can be determined by eq. (8) of the IV Chapter and few other facts e.g. $\frac{n}{h_c}$ which givs the number of one-dimensional irreducible representations where h_c is the order of the commutator subgroup. The values of c_1, c_2 ....c_r are given by eq. (10) of IV Chap.

$$c_m = \frac{1}{n} \sum_{i=1}^{n} k(g_i)\, k_m(g_i) = \frac{1}{n} \sum_{l=1}^{r} k(K_l)\, k_{ml}\, r_l$$

where $k(g_i)$ is the character of g_i in R, k_{ml} is the character of the class K_l in mth irreducible representation, r_l is the number of members in the class K_l.

Thus for the example of Fig. (3.9), the symmetry group of the system is c_{4v} and there are 16 degrees of freedom in the system. There are 5-classes in the group i.e. r = 5. The dimensionality of the irreducible representations are 1,1,1,1, and 2 i.e. $d_1 = d_2 = d_3 = d_4 = 1$, $d_5 = 2$. The characters of the 5-classes in the 16-dimensional representation in the basis shown in Fig. 3.9 are 16, 0,0 ,0,0. (The 16-dimensional representations will be found very soon). Therefore

$$c_m = \frac{1}{8} \quad k(K_l)\, k_{ml} = \frac{k(E)\; k_m(E)}{8} = 2k_m(E)$$

Now $k_m(E)$ are 1, 1, 1, 1, and 2. Therefore,

$c_1 = c_2 = c_3 = c_4 = 2$, $c_5 = 4$.

Therefore the stiffness matrix will break into 4 blocks of 2 x 2 and two identical blocks of 4 x 4, corresponding to R_5 which is two dimensional. Therefore the inversion of the 16 x 16 stiffness matrix will boil down to the inversions of four 2 x 2 matrices and one 4 x 4 matrix while the procedures of Chapter 3 (section 3.2 and 3.3) theoretically required for this example, the inversions of four 4 x 4 matrices. One can see the kind of resulting simplifications due to use of full symmetry.

Similarly the vibration and buckling problems also get simplified provided the mass matrices and axial loads in the members have same symmetries as that of structure. At this point it should be noted that for the vibration problem, the mass matrices will automatically be similar to stiffnes matrices. This follows from the definition of mass matrices. Mass matrices are defined through kinetic energy, i.e. if, $T = (1/2)\ \dot{X}^T M\dot{X} = T(\dot{X})$, then M is the mass matrix. Since the system has symmetry group G with representation $R = \{D(g_i),\ i=1, \ldots\ n\}$, therefore $T(R\dot{X}) = T(\dot{X})$ which implies

$$MR = RM \qquad \cdots\cdots\cdots (5)$$

The eq. (5) is exactly similar to eq. (4) for stiffness matrices therefore all the arguments about stiffness matrices hold true for mass matrices also and hence the mass matrices of symmetrical structural systems will have exactly similar block forms as that of the stiffness matrices (here it has been assumed that there are no other inertia sharing masses except that of the structures.)

Comming to some what more specific case consider the systems with Abelian symmetry groups. In this case all the irreducible representations are one dimensional and and hence there are n-irreducible representations R_1, R_2, ... R_n. Let R be the N-dimensional representation for the system and let R be reduced to,

$$R = c_1R_1 \oplus c_2 R_2 \oplus \ldots\ldots \oplus c_nR_n$$

Then the stiffness matrix will break into n distinct blocks of dimensions $c_1 \times c_1$, $c_2 \times c_2$ .... $c_n \times c_n$. In such problems one has to invert n-matrices of $c_1 \times c_1$, $c_2 \times c_2$, ... $c_n \times c_n$. The values of c_1, c_2, ... c_n depend upon the representation R of the problem. However their values are approximately same for every big problem.

5.2.1 Effect of Increasing or Decreasing the Symmetries of Structural Systems

It has been seen in the IV Chapter that symmetry

elements of structural systems always form a group. Thus, if a structural system has a symmetry group G and if one removes some symmetry elements or element, the rest of the symmetry elements must form another group $G_1 \subset G$. Thus removal of symmetry elements is restricted. Similarly if one wants to add some more symmetry elements or element, the resulting collection of symmetry elements must form another Group $G_2 \supset G$. This is the restriction on increasing the symmetry. If G, G_1 and G_2 all are Abelian, there will not be any any basic change in the nature of the problem except for the decrease or increase in simplifications of the problem. But if G is Abelian and G_2 is non-Abelian then some of the irreducible representations of the group G_2 will be at least two-dimensional. In this case the form of the stiffness matrix will change and some blocks will occur more than once. Which will not only result in more simplifications but also in degeneracies if one considers the vibration or buckling problem, a ... fundamental change in problem. If however G was non-Abelian and G is Abelian then in addition to decrease in simplifications, the degeneracies will be removed. (Remembering that degenracies mean only those which result from symmetry and not those due to particular choice of variables). If G_1, G and G_2 are all non-Abelian, the

result is not very specific and depends upon the type of problems etc. As an example, consider a system with the symmetry group C_3. Let the system is symmetriesed to C_{3v}. In the original system there was no trace for degeneracies while in the modified system the degeneracy is bound to occur. To be more specific, if the spring mass system of Fig. 3.10(a) is considered to have the group C_3, there will not be any indication about degeneracies in frequency, however if the same is considered to have the group C_{3v}, two of the frequencies will be degenerate because the same block of 2 x 2 in stiffness matrix will occur twice. Similarly if the system shown in Fig. 3.9 had C_{2n} or C_4, all the 4 blocks in the stiffness matrix would have been distinct and 4 x 4, but if one increases the symmetry from C_{2n} or C_4 one will end in C_{4v}, a non-Abelian group which has 5-irreducible representations, 4 of which are one dimensional and one of them is two-dimensional and hence the stiffness matrix for these problems would break into four 2 x 2 blocks and two 4 x 4 identical blocks. Therefore there will be 4 frequencies doubly degenerate (in vibration problems).

So far it has been assumed that the symmetry adapted basis has been chosen. In actual problems however, the finding of this basis itself remains a problem. The initial choice of basis plays very important role in the

sense that it eases the representation matrices. Two natural choices of the initial basis vectors are following:

(i) All the nodal co-ordinate systems are orthogonal and similarly oriented.

(ii) The nodal co-ordinate systems are so oriented that the maximum number of them coincide with each other under the symmetry operations by a maximal sub-group of the group. This choice is very very helpful because the representation matrices have very simple structure and therefore the symmetry adapted basis can be easily found.

5.3 THE RESULTS OF THE III CHAPTER AND GENERALIZATION BY USING GROUP THEORY

The results of III Chapter or for that matter the results of all earlier authors (20, 21, 22, 23) are special cases of the present result in the following sense:

(i) that here the nodes of the structural systems may be on the symmetry planes symmetry axes (or on the symmetry elements),

(ii) that various symmetries whose groups have same irreducible representations are treated at a stretch,

(iii) The orientations of individual vectors in the initial basis can be arbitrary.

5.3.1 The Structural Systems with the Symmetry Groups C_{1h}, C_2 or S_2

The groups C_{1h}, C_2 and S_2 have same irreducible

representations (please see the appendix) and therefore these symmetries produce similar simplifications. (The symmetry adapted basis, however, may be different for the symmetry groups C_{1h}, C_2 and S_2). Therefore the results derived for C_{1h} are valid for C_2 and S_2 also in their corresponding symmetry adapted basis. Due to this, in the following, only C_{1h} is considered. Note that $C_{1h} = \{E, \sigma\}$[15] representing the mirror symmetry. There are two cases in this symmetry:

CASE I: When the symmetry plane does not contain nodes.

CASE II: When the symmetry plane does contain a number of nodes.

These cases for the groups C_2 and S_2 are those in which the symmetry elements C_2 and i do not contain any nodes or contain some nodes where this C_2 is the axis of symmetry and is the inversion center.

CASE I:

In this case the system must have an even number of degrees of freedoms, the half of which are on one side of the symmetry plane and the rest half are on the other side of the symmetry plane.

Let the initial co-ordinate systems for the two sides be chosen as in section 3.1. Let there be N degrees of freedom. The basis then is, $e = (e_1, e_2 \ldots\ldots e_{N/2}, e_{\frac{N}{2}+1} \ldots\ldots\ldots e_N)$. Let $K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$ be the stiffness

matrix of the system, where K_{ij} are $(N/2) \times (N/2)$ matrices and $K_{ij}^T = K_{ji}$, in this basis.

The N-dimensional representation for this group is $R = \{D(E), D(\sigma)\}$: $D(E)$ is $N \times N$ identity matrix. The $D(\sigma)$ is found by finding the effect of on e. At this stage different kinds of numberings of basis vectors are possible and thus different matrices for representation R are possible. But the different kinds of numbering of basis vectors will not affect the final result because the resulting representation matrices will be equivalent to each other (equivalent in the sense of IV Chapter). Let us first take that numbering in which $\sigma e_1 = e_{\frac{N}{2}+1}$, $\sigma e_2 = e_{\frac{N}{2}+2}$ ....

$\ldots \sigma e_{\frac{N}{2}} = e_N$, then $D(\sigma) = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ and thus the $k(\sigma)=0$ and $k(E) = N$. Therefore if R is reduced to $R = c_1R_1 \oplus c_2R_2$ where R_1, R_2 are irreducible representations of C_{1h} and

where c_1, c_2 are given by eq. (10) of the IV Chapter, i.e.

$$c_m = \frac{1}{2n} \sum_{i=1}^{n} k(g_i)\, k_m(g_i) = \frac{1}{2}\, k(E)\, k_m(E) = (N/2)$$

$$(k(\sigma) = 0,\ k(E) = N)$$

Thus $c_1 = c_2 = (N/2)$ and so R reduces to $(N/2)R_1 \oplus \frac{N}{2} R_2$ i.e

$$D(E) = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \text{ and } D(\sigma) = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$$

where I is $(N/2) \times (N/2)$ identity matrix, in the symmetry adapted basis. The eq. (4), i.e. $KR = RK$ means now (where $\bar{K}$ is in symmetry adapted basis),

$$\begin{bmatrix} \bar{K}_{11} & \bar{K}_{12} \\ \bar{K}_{21} & \bar{K}_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \bar{K}_{11} & \bar{K}_{12} \\ \bar{K}_{21} & \bar{K}_{22} \end{bmatrix}$$

which implies

or $\bar{K}_{12} = \bar{K}_{21} = 0$ i.e. $K = \begin{bmatrix} \bar{K}_{11} & 0 \\ 0 & \bar{K}_{22} \end{bmatrix}$

However, this much is not enough, $\bar{K}_{11}$, $\bar{K}_{22}$ must be known interms of K_{ij} and therefore the knowledge of symmetry adapted basis is needed.

From eq. (11) of IV Chapter the projection operators are given by,

$$P_{ij}^{(k)} = \sum_{l=1}^{n} D_{ij}(g_l)\, g_l$$ which for C_{1h} reduces to

$$P^{(1)} = E + \sigma, \text{ and } P^{(2)} = E - \sigma$$

Therefore the symmetry adapted basis is the set of N linearly independent vectors from, $(E + \sigma)\, e_1$,

$(E + \sigma)\, e_2 \ldots\ldots (E+\sigma)\, e_N$ and $(E - \sigma)\, e_1$, $(E - \sigma)\, e_2 \ldots\ldots$ $(E - \sigma)\, e_N$. These are $\bar{e}_1 = \frac{1}{\sqrt{2}}\,(e_1 + e_{\frac{N}{2}+1})$,

$\bar{e}_2 = \frac{1}{\sqrt{2}}\,(e_2 + e_{\frac{N}{2}+2})$, ... $\bar{e}_{\frac{N}{2}} = \frac{1}{\sqrt{2}}\,(e_{\frac{N}{2}} + e_N)$,

$$\bar{e}_{\frac{N}{2}+1} = \frac{1}{\sqrt{2}}(e_{\frac{N}{2}+1} - e_1), \quad \bar{e}_{\frac{N}{2}+2} = \frac{1}{\sqrt{2}}(e_2 - e_{\frac{N}{2}+2}), \quad \ldots\ldots \quad \bar{e}_N = \frac{1}{\sqrt{2}}(e_{\frac{N}{2}} - e_N)$$

...(6)

where $\frac{1}{\sqrt{2}}$ is a normalising factor.

ie.

$$\bar{e} = \begin{bmatrix} \bar{e}_1 \\ \bar{e}_2 \\ \cdot \\ \cdot \\ \cdot \\ \bar{e}_N \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ I & -I \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \cdot \\ \cdot \\ \cdot \\ e_N \end{bmatrix} \quad \text{or } \bar{e} = S\,e.$$

Under this change of basis, the stiffness matrix gets transformed to $SKS^{-1} = SKS$ which should be nothing but that obtained by applying the schur lemmas i.e.

$$SKS = \frac{1}{2}\begin{bmatrix} I & I \\ I & -I \end{bmatrix}\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}\begin{bmatrix} I & I \\ I & -I \end{bmatrix} = \bar{K} = \begin{bmatrix} \bar{K}_{11} & 0 \\ 0 & \bar{K}_{22} \end{bmatrix}$$

$$\text{or } \frac{1}{2}\begin{bmatrix} K_{11}+K_{22}+K_{12}+K_{22} & K_{11}-K_{22}-K_{12}+K_{21} \\ K_{11}-K_{22}+K_{12}-K_{21} & K_{11}+K_{22}-K_{12}-K_{21} \end{bmatrix}$$

$$= \begin{bmatrix} \bar{K}_{11} & 0 \\ 0 & \bar{K}_{22} \end{bmatrix} \quad \text{which means that,}$$

$$K_{12} - K_{22} \pm (K_{12} - K_{21}) = 0 \quad \text{and} \quad \begin{aligned} \bar{K}_{11} &= \tfrac{1}{2}(K_{11}+K_{22}+K_{12}+K_{21}) \\ \bar{K}_{22} &= \tfrac{1}{2}(K_{11}+K_{22}-K_{12}-K_{22}) \end{aligned}$$

This verifies the form of the stiffness matrix obtained in section 3.1, i.e. $K = \begin{matrix} A & B \\ B & A \end{matrix}$ under the condition stated there. The present method indicates that the stiffness matrix will have same form even for the symmetry group C_2 and S_2 provided the initial basis is so chosen that half of the basis vectors coincide with the other half under C_2 and i respectively.

In the basis $\bar{e} = (\bar{e}_1, \bar{e}_2 \ldots . \bar{e}_N)$ one has to invert now two matrices $\bar{K}_{11}$ and $\bar{K}_{22}$ which are now $(N/2) \times (N/2)$ instead of one matrix of $N \times N$. The solution vector in the original basis is at once obtained by applying $S^{-1} = S$ to the solution vector obtained in symmetry adapted basis i.e. by the additions and substractions of proper components of solution vector in this basis and multiplied by $\frac{1}{2}$.

The corollaries of section 3.1 also follow from this result by similar arguments as in section 3.1. Here, however, it has been shown (in section 5.2) that the mass matrices of the structural systems will have exactly similar forms as that of their stiffness matrices.

If the numbering of the basis vectors is such that $\sigma e_1 = e_N$, $\sigma e_2 = e_{N-1} \ldots . , \sigma e_{\frac{N}{2}} = e_{\frac{N}{2}+1}$ then the N-dimensional representation matrix $D(\sigma)$ is given by,

$$D(\sigma) = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{bmatrix} = C \text{ the counter identity.}$$

The reduced representation of R is again given by $R = (N/2)\ R_1 \oplus (N/2)\ R_2$ because in this case also, $k(\sigma) = 0$, $k(E) = N$. This means that in symmetry adapted basis the matrices $D(E)$ and $D(\sigma)$ have following forms:

$$D(E) = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \text{ and } D(\sigma) = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$$ ($I = (N/2) \times (N/2)$ identity matrix).

This implies that the stiffness matrix will again reduce to the form $\bar{K} = \begin{bmatrix} \bar{K}_{11} & 0 \\ 0 & \bar{K}_{22} \end{bmatrix}$ in the symmetry adapted basis. However the symmetry adapted basis will be different from that of earlier case and will be given by basis vectors

$$\bar{e}_1 = \frac{1}{\sqrt{2}}(e_1 + e_N)\quad \bar{e}_2 = \frac{1}{\sqrt{2}}(e_2 + e_{N-1}), \quad \ldots \quad \bar{e}_{\frac{N}{2}} = \frac{1}{\sqrt{2}}(e_{\frac{N}{2}} + e_{\frac{N}{2}+1}),$$

$$\bar{e}_{\frac{N}{2}+1} = \frac{1}{\sqrt{2}}(e_1 - e_N), \quad \bar{e}_{\frac{N}{2}+2} = \frac{1}{\sqrt{2}}(e_2 - e_{N-1}) \quad \ldots \quad \bar{e}_N = \frac{1}{\sqrt{2}}(e_{\frac{N}{2}} - e_{\frac{N}{2}+1}).$$

Thus the symmetry adapted basis $\bar{e}$ is obtained by following transformation

$$\bar{e} = \begin{bmatrix} \bar{e}_1 \\ \bar{e}_2 \\ \vdots \\ \bar{e}_N \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & & & & & 1 \\ & 1 & & & 1 & \\ & & 1 & \cdot & & \\ & & \cdot & \cdot & & \\ & \cdot & & & \cdot & \\ 1 & & & & & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_N \end{bmatrix} \text{ or } \bar{e} = \begin{bmatrix} I & C \\ C & -I \end{bmatrix} e = se$$

The corresponding transformation of stiffness matrix is given by

$$\bar{K} = SKS^{-1} = \frac{1}{2}\begin{bmatrix} I & C \\ C & -I \end{bmatrix}\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}\begin{bmatrix} I & C \\ C & -I \end{bmatrix} = \begin{bmatrix} \bar{K}_{11} & 0 \\ 0 & \bar{K}_{22} \end{bmatrix}$$

or
$$\begin{bmatrix} K_{11}+CK_{22}C + K_{12}C + CK_{21} & K_{11}C-CK_{22}-K_{12} +CK_{21}C \\ CK_{11}-K_{22} +CK_{12}C - K_{21} & CK_{11}C+ K_{22}-CK_{12}-K_{21}C \end{bmatrix} = \begin{bmatrix} \bar{K}_{11} & 0 \\ 0 & \bar{K}_{22} \end{bmatrix}$$

i.e. $K_{11}C-CK_{22}-K_{12} +CK_{21}C=0$ which can be true only if K is centro-symmetric ie. if

$$\begin{bmatrix} 0 & C \\ C & 0 \end{bmatrix}\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}\begin{bmatrix} 0 & C \\ C & 0 \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

$$(K_{ij} = K^T_{ji}\ ,\ i,\ j = 1,\ 2)$$

or if $CK_{21}C = K_{12}$, $CK_{22}C = K_{11}$. This shows that the stiffness matrix is centro-symmetric in the original basis, and in the symmetry adapted basis

$$K = \begin{bmatrix} \bar{K}_{11} & 0 \\ 0 & \bar{K}_{22} \end{bmatrix} \quad \text{where} \quad \bar{K}_{11} = K_{11} + K_{12}\,C;\ \bar{K}_{22} = K_{22} - CK_{12}$$

This illustrates the fact that numbering of basis vectors does not matter as for as the simplification, of the problem is concerned. The stiffness matrix is bound to be block diagonal if the symmetry adapted basis is chosen.

Comming to the orientations of basis vectors, it is also immaterial. However, as remarked earlier, a choice of proper orientation eases the problem of getting

symmetry adapted basis and the subsequent block form stiffness matrix etc. As an illustration, consider the above case but with such orientations of e_1, e_2 .... e_N, that $\sigma e_1 = - e_{\frac{N}{2}+1}$, $\sigma e_2 = - e_{\frac{N}{2}+2}$, ... $\sigma e_{\frac{N}{2}} = - e_N$.

The N-dimensional representation of the group is now $R = \{D(E), \quad D(\sigma)\}$ where $D(E)$ = N x N identity matrix and

$$D(\sigma) = \begin{bmatrix} 0 & -I \\ -I & 0 \end{bmatrix}$$

and once again the reduced representation is given by $R = (N/2)\ R_1 \oplus (N/2)R_2$ which implies the same kinds of blocks in K is in the symmetry adapted basis.

The symmetry adapted basis is generated by the two projection operators $P^{(1)} = E + \sigma$ and $P^{(2)} = E - \sigma$ and is given by N - linearly independent vectors out of,

$$(E + \sigma)e_1, \quad (E + \sigma)\ e_2\ .....\ (E+\sigma)e_N, (E - \sigma)e_1\ ....\ (E-\sigma)e_N$$

and is given by vectors

$$e_1 = \frac{1}{\sqrt{2}}(e_1 + e_{\frac{N}{2}+1}), \quad \bar{e}_2 = \frac{1}{\sqrt{2}}(e_2 + e_{\frac{N}{2}+2})\ ....\ \bar{e}_{\frac{N}{2}} = \frac{1}{\sqrt{2}}(e_{\frac{N}{2}} + e_N),$$

$$\bar{e}_{\frac{N}{2}+1} = \frac{1}{\sqrt{2}}(e_1 - e_{\frac{N}{2}+1}), \quad \bar{e}_{\frac{N}{2}+2} = \frac{1}{\sqrt{2}}(e_2 - e_{\frac{N}{2}+2}) \ldots \bar{e}_N = \frac{1}{\sqrt{2}}(e_{\frac{N}{2}} - e_N) \qquad (7)$$

This basis is same as given in eq. (6), therefore every thing that followed after eq. (6) remains valid for this case also.

Therefore, numbering or orientations of original basis vectors is immaterial so far as simplification resulting

from symmetry is concerned except the side labour which is needed due to inconvenient original basis.

CASE II

Let 'm' degrees of freedom are coming on the reflection plane . If N is the total number of degrees of freedom then N-m must be even because of symmetry, i.e. (1/2)(N-m) degrees of freedom will be on one side of σ and the rest $\frac{1}{2}(N-m)$ will be on the other side of σ. Let the original basis be given by vectors,

$$e_1, e_2 \ldots\ldots e_{\frac{N-m}{2}}, \quad e_{\frac{N-m}{2}+1} \ldots\ldots e_{\frac{N+m}{2}}, e_{\frac{N+m}{2}+1} \ldots e_N.$$

Let these be so oriented and numbered that,

$$\sigma e_1 = e_{\frac{N+m}{2}+1}, \quad \ldots\ldots\ldots \sigma e_{\frac{N-m}{2}} = e_N \quad \text{and}$$

$$\sigma e_{\frac{N-m}{2}+1} = -e_{\frac{N-m}{2}+1} \quad \ldots\ldots \sigma e_{\frac{N-m}{2}+m_1} = -e_{\frac{N-m}{2}+m_1}, \quad \sigma e_{\frac{N-m}{2}+m_1+1}$$

$$= e_{\frac{N-m}{2}+m_1+1} \quad \ldots\ldots\ldots \sigma e_{\frac{N+m}{2}} = e_{\frac{N+m}{2}}$$

Then the N-dimensional representation R is given by D(E) = NxN identity matrix and,

$$D(\sigma) = \begin{bmatrix} & & & I \\ & -I_1 & & \\ & & I_2 & \\ I & & & \end{bmatrix}$$

Where I, I_1 and I_2 are $\frac{N-m}{2} \times \frac{N-m}{2}$, $m_1 \times m_1$ and $\frac{m-m_1}{2} \times \frac{m-m_1}{2}$ identity matrices, respectively.

The characters of R are $k(E) = N$, $k(\sigma) = 2m - m_1$

Now if R is reduced to $R = c_1 R_1 \oplus c_2 R_2$ where $c_1 = \frac{1}{2}(N+2m-m_1)$ and $c_2 = \frac{1}{2}(N-2m + m_1)$ or $c_1 = \frac{1}{2}(N+m) + (1/2)(m-m_1)$ and $c_2 = \frac{1}{2}(N-m) - \frac{1}{2}(m-m_1)$ determined from eq. (10) of IV Chapter. Since c_1 and c_2 are positive integers and $\frac{1}{2}(N \pm m)$ is a positive integer therefore $m-m_1$ must be even or zero.

Therefore, the stiffness matrix will now break into blocks of $\frac{1}{2}(N-2m+m_1) \times \frac{1}{2}(N-2m + m_1)$ and $\frac{1}{2}(N+2m-m_1) \times \frac{1}{2}(N+2m-m_1)$ and the arguments of symmetry are applicable. To be specific the following example will be considered.

Example: Consider a spring mass system (Fig. 3.1.b) with the symmetry group C_{1h}. The system has 2N+1 degrees of freedom with the N+1th mass coming on the reflection plane. The basis vectors are as shown in Fig.(3.1.b).The effect of this on basis vectors is as follows:

$\sigma e_1 = e_{N+2}$, $\sigma e_2 = e_{N+3}$, $\ldots \sigma e_N = e_{2N+1}$ and $\sigma e_{N+1} = -e_{N+1}$.

Therefore the 2N+1 dimensional representation of C_{1h} is given by matrices, $D(E) = 2N+1 \times 2N+1$ identity matrix and

$$D(\sigma) = \begin{bmatrix} & & I \\ & -1 & \\ I & & \end{bmatrix}$$ where I is N x N identity matrix.

Therefore the characters for R and $k(E) = 2N+1$ and $k(\sigma) = -1$

Therefore R is reduced to $R = c_1 R_1 \oplus c_2 R_2$ where c_1, c_2 are given by eq. (10) of IV Chapter and are $\frac{1}{2}(2N+1 -1)$ and

$\frac{1}{2}$ (2N+1+1) i.e. N and N+1 respectively, i.e

$$R = NR_1 \oplus (N+1) R_2, D(\sigma) = \begin{bmatrix} I & 0 \\ 0 & -I_1 \end{bmatrix}$$ where I and I_1 are N x N and N+1 x N+1 identity matrices respectively.

Therefore in symmetry adapted basis the stiffness and mass matrices will break to N x N and N+1 x N+1 blocks. The symmetry adapted basis $\bar{e}$ for this case is found to be given by vectors, $\frac{1}{\sqrt{2}}(e_1 + e_{N+2}), \frac{1}{\sqrt{2}}(e_2 + e_{N+3}) \ldots\ldots$

$\ldots\ldots \frac{1}{\sqrt{2}}(e_N + e_{2N+1}), \quad e_{N+1}, \quad \frac{1}{\sqrt{2}}(e_1 - e_{N+2}), \frac{1}{\sqrt{2}}(e_2 - e_{N+3}) \ldots$

$\ldots\ldots \frac{1}{\sqrt{2}}(e_N - e_{2N+1})$ (7)

The above is obtained by eq. (11) and (12) of IV Chapter which in here is $(E \pm \;) e_i$ (i=1,2, ... 2N+1). The transformation matrix associated with this symmetry adapted bassis is given by $\bar{e}$ = se i.e.

$$\bar{e} = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_{2N+1} \end{bmatrix} = \left[\begin{array}{ccc} \frac{1}{\sqrt{2}} I & & \frac{1}{\sqrt{2}} I \\ \hline & 1 & \\ \hline \frac{1}{\sqrt{2}} I & & -\frac{1}{\sqrt{2}} I \end{array}\right] \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_{2N+1} \end{bmatrix}$$

Under the transformation SKS^{-1} = SKS, the original stiffness matrix K gets transformed to block diagonal matrix of N x N and N+1 x N+1 blocks. The original stiffness matrix K will have the form,

$$K = \begin{bmatrix} A & b & 0 \\ b^T & a & -b^T \\ 0 & -b & A \end{bmatrix}$$

where b is a colum matrix, a is a number and A is N x N matrix.

$$\text{and} \quad SKS = \begin{bmatrix} A & 0 & 0 \\ 0 & a & \sqrt{2}b^T \\ 0 & \sqrt{2}b & A \end{bmatrix}$$

The mass matrix is diag. $(m_1\ m_2\ \ldots\ m_{2N+1})$. Therefore the natural frequencies are given by two determinants one of order N and the other of order N+1. Thus forthe 5-masses & 6-springs system of Fig. 2.9, stiffness matrix

$$K = \begin{bmatrix} k_1+k_2 & -k_2 & 0 & 0 & 0 \\ -k_2 & k_2+k_3 & -k_3 & 0 & 0 \\ 0 & -k_3 & 2k_3 & 0 & k_3 \\ 0 & 0 & 0 & k_1+k_2 & -k_2 \\ 0 & 0 & k_3 & -k_2 & k_2+k_3 \end{bmatrix} \qquad M = \begin{bmatrix} m_1 & & & & \\ & m_2 & & & \\ & & m_3 & & \\ & & & m_1 & \\ & & & & m_2 \end{bmatrix}$$

$$SKS = \begin{bmatrix} k_1+k_2 & -k_2 & & & \\ -k_2 & k_2+k_3 & & & \\ & & 2k_3 & 0 & -2^{\frac{1}{2}}k_3 \\ & & 0 & k_1+k_2 & -k_2 \\ & & -\sqrt{2}k_3 & -k_2 & k_2+k_3 \end{bmatrix} \quad \text{and } SMS = M$$

Therefore the frequencies are given by,

$$\det \begin{vmatrix} k_1+k_2 -m_1w^2 & -k_2 \\ -k_2 & k_2+k_3-m_2w^2 \end{vmatrix} = 0$$

$$\text{and } \det \begin{vmatrix} 2k_3-m_3w^2 & 0 & -\sqrt{2}k_3 \\ 0 & k_1+k_2-m_1w^2 & -k_2 \\ -\sqrt{2}k_3 & -k_2 & k_2+k_3- mw^2 \end{vmatrix} = 0$$

which can be solved in no time.

Thus group theory is able to use symmetry in all situations and grinds the stiffness matrix to block forms irrespective of the initial choice and numberings of the co-ordinate systems.

5.3.2 The Structural Systems With the Symmetry Groups C_{2v}, C_{2h} and D_2

Section 3.2 of III Chapter was exclusively devoted to the symmetrical structures with C_{2v} where this type of symmetry was called "the double orthogonal reflection plane symmetry" and where the symmetry element C_2 was not used. That procedure is not applicable if some of the nodes are situated on the reflection planes σ_1 or σ_2. The procedure also required definite types of four global co-ordinate systems. Group theory removes all these deficiencies and deals with different kinds of symmetries given by C_{2v}, C_{2h} and

D_2 at one stretch. Two cases, one corresponding to no nodes on symmetry elements and the other in which nodes are on symmetry elements also, are discussed separately. In the following only C_{2v} is considered and the cases with C_{2h} or D_2 go hand in hand with this except for the differences of symmetry elements.

CASE I

In this case the number of degrees of freedom must be devisible by 4 i.e. $\frac{N}{4}$ is an integer. Let initial basis vectors be,

$$e_1, e_2 \ldots\ldots e_{\frac{N}{4}}, \quad e_{\frac{N}{4}+1} \ldots\ldots e_{\frac{N}{2}}, \ e_{\frac{N}{2}+1} \ldots\ldots e_{\frac{3N}{4}}, \ e_{\frac{3N}{4}+1} \ldots e_N.$$

Let they be so oriented and numbered that,

$$\sigma_1 \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_{\frac{N}{4}} \end{bmatrix} = \begin{bmatrix} e_{\frac{N}{4}+1} \\ e_{\frac{N}{4}+2} \\ \vdots \\ e_{\frac{N}{2}} \end{bmatrix}, \quad \sigma_1 \begin{bmatrix} e_{\frac{N}{2}+1} \\ e_{\frac{N}{2}+2} \\ \vdots \\ e_{\frac{3N}{4}} \end{bmatrix} = \begin{bmatrix} e_{\frac{3N}{4}+1} \\ e_{\frac{3N}{4}+2} \\ \vdots \\ e_N \end{bmatrix}$$

and

$$\sigma_2 \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_{\frac{N}{4}} \end{bmatrix} = \begin{bmatrix} e_{\frac{N}{2}+1} \\ e_{\frac{N}{2}+2} \\ \vdots \\ e_{\frac{3N}{4}} \end{bmatrix}, \quad \sigma_2 \begin{bmatrix} e_{\frac{N}{4}+1} \\ e_{\frac{N}{4}+2} \\ \vdots \\ e_{\frac{N}{2}} \end{bmatrix} = \begin{bmatrix} e_{\frac{3N}{4}+1} \\ e_{\frac{3N}{4}+2} \\ \vdots \\ e_N \end{bmatrix}$$

and noting that $\sigma_1^2 = \sigma_2^2 = E$ and $\sigma_1 \sigma_2 = \sigma_2 \sigma_1 = C_2$, the N-dimensional representation 'R' of C_{2v} is given by

$$D(E) = \begin{bmatrix} I & & & \\ & I & & \\ & & I & \\ & & & I \end{bmatrix} \quad D(\sigma_1) = \begin{bmatrix} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{bmatrix},$$

$$D(\sigma_2) = \begin{bmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix} \quad \text{and } D(C_2) = \begin{bmatrix} & & & I \\ & & I & \\ & I & & \\ I & & & \end{bmatrix}$$

where I is $\frac{N}{4} \times \frac{N}{4}$ identity matrix. Since the stiffness matrix must remain in-variant under R, it will assume the form shown in section 3.2. The irreducible representations for these groups are given in Appendix Table A-2. Let the reduced representation of R be $R = c_1R_1 \oplus c_2R_2 \oplus c_3R_3 \oplus c_4R_4$ where $c_1 = c_2 = c_3 = c_4 = \frac{N}{4}$ which was found in Chapter 4. This shows that the stiffness matrix will break into 4 blocks of $\frac{N}{4} \times \frac{N}{4}$ provided symmetry adapted basis is chosen.

The projection operators for C_{2v} are $P^{(1)} = E + C_2 + \sigma_1 + \sigma_2$, $P^{(2)} = E + C_2 - \sigma_1 - \sigma_2$, $P^{(3)} = E - C_2 + \sigma_1 - \sigma_2$ and $P^{(4)} = E - C_2 - \sigma_1 + \sigma_2$. (Here the 4 irreducible representations are one-dimensional and thus eq. (11) of IV Chapter should be use with no suffix on $P_{ij}^{(k)}$). The symmetry adapted basis is given (by using eq. (12) of IV Chapter), by, $P^{(k)}e_i$ (k=1,2,3,4, i=1,2, ... N) and are N linearly

normalised vectors out of these. Recalling the effects of σ_1, σ_2 and C_2 on initial basis vectors one has the symmetry adapted basis vectors as

$$\bar{e}_1 = \frac{1}{2}(e_1 + e_{\frac{N}{4}+1} + e_{\frac{N}{2}+1} + e_{\frac{3N}{4}+1}) \;\ldots\; \bar{e}_{\frac{N}{4}} = \frac{1}{2}(e_{\frac{N}{4}} + e_{\frac{N}{2}} + e_{\frac{3N}{4}} + e_N)$$

$$\bar{e}_{\frac{N}{4}+1} = \frac{1}{2}(e_1 - e_{\frac{N}{4}+1} + e_{\frac{N}{2}+1} - e_{\frac{3N}{4}+1}) \;\ldots\; \bar{e}_{\frac{N}{2}} = \frac{1}{2}(e_{\frac{N}{4}} - e_{\frac{N}{2}} + e_{\frac{3N}{4}} - e_N)$$

$$\bar{e}_{\frac{N}{2}+1} = \frac{1}{2}(e_1 + e_{\frac{N}{4}+1} - e_{\frac{N}{2}+1} - e_{\frac{3N}{4}+1}) \;\ldots\; \bar{e}_{\frac{3N}{4}} = \frac{1}{2}(e_{\frac{N}{4}} + e_{\frac{N}{2}} - e_{\frac{3N}{4}} - e_N)$$

$$\bar{e}_{\frac{3N}{4}+1} = \frac{1}{2}(e_1 - e_{\frac{N}{4}+1} - e_{\frac{N}{2}+1} + e_{\frac{3N}{4}+1}) \;\ldots\; \bar{e}_N = \frac{1}{2}(e_{\frac{N}{4}} - e_{\frac{N}{2}} - e_{\frac{3N}{4}} + e_N)$$

or

$$\bar{e} = \begin{bmatrix} \bar{e}_1 \\ \bar{e}_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \bar{e}_N \end{bmatrix} = \frac{1}{2} \begin{bmatrix} I & I & I & I \\ I & -I & I & -I \\ I & I & -I & -I \\ I & -I & -I & I \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ e_N \end{bmatrix} = Se \text{ and } S^{-1} = S^T = S$$

I is $\frac{N}{4} \times \frac{N}{4}$ identity matrix.

The corresponding transormed stiffness matrix in this new basis is $\bar{K} = SKS$. Since this basis is symmetry adapted therefore K will be block diagonal, each block of $\frac{N}{4} \times \frac{N}{4}$, i.e.

$$\overline{K} = \begin{bmatrix} \overline{K}_{11} & & & \\ & \overline{K}_{22} & & \\ & & \overline{K}_{33} & \\ & & & \overline{K}_{34} \end{bmatrix} \quad \text{where} \quad \begin{aligned} \overline{K}_{11} &= K_{11} + K_{12} + K_{13} + K_{14} \\ \overline{K}_{22} &= K_{11} - K_{12} + K_{13} - K_{14} \\ \overline{K}_{33} &= K_{11} + K_{12} - K_{13} - K_{14} \\ \overline{K}_{44} &= K_{11} - K_{12} - K_{13} + K_{14} \end{aligned}$$

where K_{ij} are the partioned sub-matrioes of K in original basis which are equivalent to the matrices A, B, C, D of the section 3.2. Therefore the solution of the problem in the symmetry adapted basis $\overline{e}$ is found simply by inverting the four (N/4) x (N/4) matrices $\overline{K}_{jj}$ (j=1,2,3,4). This result tallies with that of III Chapter (section 3.2) because the initial choice of basis is same as that of section 3.2. However, this sort of choice of the initial basis was not at all necessary here. Any arbitrary initial basis would have served the purpose and the only difference that would have come was that the representation matrices and also S and K would have been in complicated form. But these complications no more restrictone to use the symmetry arguments to simplify the problems.

To getthe solution in original basis, one has to transform the solution of the symmetry adapted basis by the matrix $S^{-1} = S^{T} = S$. Note that the matrix S is orthogonal because the symmetryadapted and initial basis vectors are always chosen to be orthogonal in N-dimensional space.

All that has ben said about the structural systems with C_{2v} is valid for the structural systems with C_{2h} or D_2. The symmetry adapted basis will of-course be different for different symmetry group, if the initial basis for all of them is same. However, if initial basis is chosen to be such that the basis vectors of one part of the structure coincide with those of other parts under symmetry operations by the symmetry elements of the corresponding groups, the symmetry adapted basis vectors may have same expression in terms of the initial basis vectors for all of the three groups and hence all of the three groups may have same 'S' and same forms of stiffness matrices in their respective basis.

CASE-II When some of the nodes come on σ_1, σ_2 or C_2, then the basis vectors corresponding to these nodes will behave differently from other basis vectors under symmetry operations and therefore they should be seperated out from the rest of the basis vectors. Let there be m such degrees of freedom (or basis vectors). Then $\frac{N-m}{4}$ must be a positive integer (from symmetry). Let the rest of the N-m basis vectors be chosen similar to those of CASE I. Then if these are,

$$e_1, e_2 \ldots\ldots e_{\frac{N-m}{4}}, e_{\frac{N-m}{4}+1} \ldots\ldots e_{\frac{N-m}{2}}, e_{\frac{N-m}{2}+1} \ldots\ldots$$

$e_{\frac{3(N-m)}{4}}$, $e_{\frac{3(N-m)}{4}+1} \ldots\ldots e_{N-m}$ then $\sigma_i \cdot e_k$ etc. are

exactly similar to case I with N-m replacing N. Let out of m basis vectors on the symmetry elements, m_1 belong to σ_1 m_2 belong to σ_2 and m_3 belong to C_2 and ;

let m_{11} and m_{12} basis vectors respectively do not change and change their signs under σ_1.

Let m_{21} and m_{12} basis vectors respectively do not change and change their signs under σ_2.

Let m_{31} and m_{32} basis vectors respectively do not change and change their signs under C_2.

One can now find out the N-dimensional representation matrices $D(E)$, $D(\sigma_1)$, $D(\sigma_2)$ and $D(C_2)$. $D(E)$ will be $N \times N$ identity matrix and $D(\sigma_1)$, $D(\sigma_2)$ and $D(C_2)$ matrices have similar forms as in CASE I for the N-m basis vectors not lying on the σ_1, σ_2 and C_2.

Once the representation is found one can immediately know the dimensionalities of the blocks in which the stiffness matrix will break. The symmetry adapted basis and the corresponding transformation matrix S can be obtained from the representation and the projection operators for the group C_{2v} which have been already found in Case I.

5.3.3 The Structural Systems with Symmetry Group: C_n:

For this type of system also, there are two cases.

CASE I: When nodes do not occur on the axis of symmetry C_n.

CASE II: When some node occurs on the axis of symmetry.

The case I was discussed in the III Chapter and there was no means to consider the II-case there. These two cases will be dealt seperately.

CASE I: Let there be N-degrees of freedom such that (N/n) = a positive iteger. Let the basis vectors $e_1 \ldots$ $, \ldots e_{\frac{N}{n}}, \; e_{\frac{N}{n}+1} \ldots e_{\frac{2N}{n}}, \; e_{\frac{2N}{n}+1} \ldots e_{\frac{3N}{n}}, \ldots\ldots e_N$ be so chosen that $C_n e_1 = e_{\frac{N}{n}+1}$, $C_n e_2 = e_{\frac{N}{n}+2} \ldots C_n e_{\frac{N}{n}} = e_{\frac{2N}{n}}$

$C_n e_{\frac{N}{n}+1} = e_{\frac{2N}{n}+1}$, $C_n e_{\frac{N}{n}+2} = e_{\frac{2N}{2}+2} \ldots C_n e_{\frac{2N}{n}} = e_{\frac{3N}{n}}$ etc.

Therefore the N-dimensional representation matrix for symmetry element C_n is,

$$D(C_n) = \begin{bmatrix} & I & & & \\ & & I & & \\ & & & \ddots & \\ & & & & I \\ I & & & & \end{bmatrix} = A_1 = T_1^T \quad \text{of section 3.3.}$$

Note that $D(C_n)$ would have been $T_1 = A_1^T$ if either the C_n was a clock-wise rotation or a different numbering of the basis vectors would have been adopted. Now it is known that all other elements of the group C_n are powers of C_n, i.e. $C_n = \{E, \; C_n, \; C_n^2, \ldots C_n^{n-1}\}$ so,

$$D(C_n^2) = D^2(C_n), \quad D(C_n^3) = D^3(C_n) \ldots D(C_n^{n-1}) = \overset{n-1}{D}(C_n)$$

and this is why in III Chapter it was found that $A_r = A_1^r$ and $T_r = T_1^r$ $(r = 1,2, \ldots\ldots n-1)$. Thus the symmetry element C_n is the generator of the cyclic group C_n. Therefore the N-dimensional representation of the group C_n is R where $R = \{E, (T_1 \text{ or } A_1), (T_1^2 \text{ or } A_1^2) \ldots\ldots (T_1^{n-1} \text{ or } A_1^{n-1})\}$. The irreducible representations of this group are given in Table 4.11. The group is Abelian and therefore every irreducible representation is one-dimensional and hence in the stiffness matrix there will be n-distinct blocks of $\frac{N}{n} \times \frac{N}{n}$ if symmetry adapted basis is chosen.

The projection operators for this group are given by

$$P_{ij}^{(k)} = \sum_{l=1}^{n} D_{kij}(g_l)\, g_l \quad \text{or} \quad P^{(k)} = \sum_{l=1}^{n} D_k(g_l)\, g_l$$

where the sufficesij have been droped. From Table 4.11 one can get now,

$$P^{(1)} = E + C_n + C_n^2 + \ldots\ldots + C_n^{n-1}$$

$$P^{(2)} = E + \theta_1 C_n + \theta_1^2 C_n^2 + \ldots\ldots\ldots + \theta_1^{n-1} C_n^{n-1}$$

$$P^{(3)} = E + \theta_2 C_n + \theta_2^2 C_n^2 + \ldots\ldots\ldots + \theta_2^{n-1} C_n^{n-1}$$

- -

$$P^{(n)} = E + \theta_{n-1} C_n + \theta_{n-1}^2 C_n^2 + \ldots\ldots + \theta_{n-1}^{n-1} C_n^{n-1}$$

The symmetry adapted basis vectors are given by eq. (12) of the IV Chapter, and any N-linearly independent normalised vectors out of $P^{(k)} e_i$ $(i=1,2, \ldots N)$ are **the set** of symmetry adapted basis vectors and are,

$$\bar{e}_1 = P^{(1)} e_1 = e_1 + e_{\frac{N}{n}+1} + e_{\frac{2N}{n}+1} + \ldots\ldots + e_{\frac{n-1}{n}N+1}$$

$$\bar{e}_2 = P^{(1)} e_2 = e_2 + e_{\frac{N}{n}+2} + e_{\frac{2N}{n}+2} + \ldots\ldots + e_{\frac{n-1}{n}N+2}$$

- -

$$\bar{e}_{\frac{N}{n}} = P^{(1)} e_{\frac{N}{n}} = e_{\frac{N}{n}} + e_{\frac{2N}{n}} + e_{\frac{3N}{n}} + \ldots\ldots\ldots + e_N$$

$$\bar{e}_{\frac{N}{n}+1} = P^{(2)} e_1 = e_1 + \theta_1 e_{\frac{N}{n}+1} + \theta_1^2 e_{\frac{2N}{n}+1} + \ldots\ldots + \theta_1^{n-1} e_{\frac{n-1}{n}N+1}$$

$$\bar{e}_{\frac{N}{n}+1} = P^{(2)} e_2 = e_2 + \theta_1 e_{\frac{N}{n}+1} + \theta_1^2 e_{\frac{2N}{n}+2} + \ldots\ldots\ldots + \theta_1^{n-1} e_{\frac{n-1}{n}N+2}$$

- -

$$\bar{e}_{\frac{2N}{n}} = P^{(2)} e_{\frac{N}{n}} = e_{\frac{N}{n}} + \theta_1 e_{\frac{2N}{n}} + \theta_1^2 e_{\frac{3N}{n}} + \ldots\ldots + \theta_1^{n-1} e_N$$

- -

- -

$$\bar{e}_N = P^{(n)} e_{\frac{N}{n}} = e_{\frac{N}{n}} + \theta_{n-1} e_{\frac{2N}{n}} + \theta_{n-1}^2 e_{\frac{3N}{n}} + \ldots\ldots + \theta_{n-1}^{n-1} e_N$$

and multiplied by a normalising factor $\frac{1}{\sqrt{n}}$.

i.e.

$$S^{-1} = \frac{1}{\sqrt{n}} \left[\begin{array}{ccccc} I & I & I & \ldots\ldots & I \\ I & I\theta_1^{n-1} & I\theta_2^{n-1} & \ldots\ldots & I\theta_{n-1}^{n-1} \\ I & I\theta_1^{n-2} & I\theta_2^{n-2} & \ldots\ldots & I\theta_{n-1}^{n-2} \\ \hline I & I\theta_1 & I\theta_2 & \ldots\ldots & I\theta_{n-1} \end{array} \right]$$

Note that the present S is nothing but T^T of section 3.3 with some modifications (note that $\theta_n=1$). Therefore all the results of III Chapter follow if the algebraic manipulations similar to that of III Chapter are carried out. The purpose here is not to repeat those, but to high light the use of the group theory. To this end one must note that if basis transorms with S then the components, (i.e. co-ordinates) will transform by S^T or S^{T*} for real or complex transformation matrices repectively. Not much of emphasis has been given to this point in previous discussions because the basis vectors have been understood to be the actual components and in fact SKS^T and S^TKS are both have been called similarity transformation.

<u>CASE II</u> Let there be m-degrees of freedoms corresponding to nodes on axis of symmetry. Then if total number of degrees of freedom is N, $\frac{N-m}{n}$ is a positive integer. Now choosing similar basis as that in Case I for the N-m

degrees of freedom which do not lie on the axis of symmetry the transformations for these basis vectors remain the same as in Case I. Let the rest of the basis vectors transform under C_n by a matrix A. Then the N-dimensional representation matrices for the symmetry elements C_n, C_n^2, ... C_n^{n-1} are,

$$D(C_n) = \begin{bmatrix} A & \\ & A \end{bmatrix} \quad D(C_n^2) = \begin{bmatrix} A_1^2 & \\ & A^2 \end{bmatrix} \cdots\cdots$$

$$\cdots\cdots D(C_n^{n-1}) = \begin{bmatrix} A_1^{n-1} & \\ & A^{n-1} \end{bmatrix}$$

where A_1, A_1^2 ..... A_1^{n-1} are the matrices A_1, A_2 .... A_{n-1} of section 3.3 and are of N-m x N-m. Note that instead of A_i one would have obtained $T_i = A_i^T$ if either C_n was a clockwise rotation or the numbering was different. Let the $tr.A^r = a_r$ then the characters of the representation are

$k(C_n) = a_1$, $k(C_n^2) = a_2$ ..... $k(C_n^{n-1}) = a_{n-1}$, $k(E)=N-m+m = N$

i.e. $tr.\ A^n = m$.

Then if this representation R is reduced to,

$R = c_1 R_1 \oplus c_2 R_2 \oplus \ldots\ldots + c_n R_n$, the values of c_1 are given by, $c_r = \frac{1}{n} \sum_{l=1}^{n} k(g_l)\, k_r(g_l)$ (r=1, 2, ...n)

or

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \frac{1}{n} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \theta_1 & \theta_1^2 & \cdots & \theta_1^{n-1} \\ 1 & \theta_2 & \theta_2^2 & \cdots & \theta_2^{n-1} \\ - & - & - & - & - \\ 1 & \theta_{n-1} & \theta_{n-1}^2 & \cdots & \theta_{n-1}^{n-1} \end{bmatrix} \begin{bmatrix} N \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

The moment one has determined the matrix A, one has found $c_1, c_2 \ldots c_n$ (which must be positive integers i.e. $N + \sum_{l=1}^{n-1} a_l \theta_r^l$ for $r = 1, 2, \ldots n$, must be devisible by n). Thus the blocks in which stiffness matrix will break now, are of $c_1 \times c_1$, $c_2 \times c_2$, ... $c_n \times c_n$ which need not be $(\frac{N-m}{n}) \times (\frac{N-m}{n})$ viz. some of them will be larger than $(\frac{N-m}{n}) \times (\frac{N-m}{n})$ some of them will be smaller than $(\frac{N-m}{n}) \times (\frac{N-m}{n})$ and some of them will even be of $(\frac{N-m}{n}) \times (\frac{N-m}{n})$.

The projection operators will remain same because they have nothing to do with the nature of problems. However symmetry adapted basis vectors will differ in those parts which are contributed by the vectors on the axis of symmetry. Thus if $\bar{e} = S\, e$ be the symmetry adapted basis then,

$$\begin{bmatrix} \bar{e}_1 \\ \cdot \\ \cdot \\ \cdot \\ \bar{e}_{N-m} \\ \cdot \\ \cdot \\ \cdot \\ \bar{e}_N \end{bmatrix} = \begin{bmatrix} S_1 & S_2 \\ & \\ S_3 & S_4 \end{bmatrix} \begin{bmatrix} e_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ e_{N-m} \\ \cdot \\ \cdot \\ \cdot \\ e_N \end{bmatrix}$$

where S_1 is S of the Case I but of $N-m \times N-m$, S_2, S_3, and S_4 are some matrices depending upon the nature of the matrix A.

Therefore the matrix A, which depends upon the problem with which one deals, must be found first and then S_2, S_3 and S_4 follow from that.

Similar analysis can be applied for the groups C_{nh} on same lines as that of C_n. The result will be that obtained in section 3.4 (Theorem 3 and 4). But group theory can also accommodate nodes on C_n or σ_h or on both.

It can be seen now that almost all the difficulties encountered in III Chapter are resolved. There is no restriction on initial choice of basis or co-ordinate systems or even numbering of various degrees of freedom. The transformation matrix which block diagonalises the stiffness matrix is found by a definite procedure which is applicable for any kind of symmetry.

5.4 HIGHER SYMMETRIES IN STRUCTURAL SYSTEMS:

Some of the higher symmetries which can not be used in full by matrix methods are those whose symmetry groups are C_{nv}, D_n or D_{nh} for $n = 3, 4, 5, \ldots,$ T, T_d, O and O_h. The most usual higher symmetries in civil engineering structures correspond to C_{nv}. However in finite element methods, one may encounter symmetries with groups D_n, D_{nh}, T, T_d etc. The present work is not concerned with finite element methods and therefore only C_{nv} will be considered. C_{2v} was already considered and at present C_{3v} and C_{4v} will be considered. To author's knowledge, the irreducible representations etc. for the general C_{nv} group are not available and therefore only particular cases will be considered here and that too with reference to a few problems which have been solved in III Chapter.

5.4.1 The Group C_{3v}

A few systems whose symmetry groups are C_{3v}, are shown in Figs. 2.1, 2.2, 2.12 and 2.13. There can be many many such systems. As discussed earlier, for these systems the stiffnes matrix will break to one block of $c_1 \times c_1$ one block of $c_2 \times c_2$ and two identical blocks of $c_3 \times c_3$ where c_1, c_2, c_3 are the number of occurrences of the three irreducible representations R_1, R_2, R_3 in the

representation of the problem at hand. To be specific, consider the example -2 of section 3.3.

Example:

Let the initial basis be chosen as that of Fig. 3.10(a) and the basis vectors be e_1, e_2, e_3, e_4, e_5, e_6. Consider the effects of the symmetry elements E, C_3, C_3^2, σ_1, σ_2 and σ_3 ($\sigma_v^a = \sigma_1$, $\sigma_v^b = \sigma_2$, $\sigma_v^c = \sigma_3$) on these basis vectors. This is given in Table 5.1.

TABLE 5.1

EFFECTS OF THE SYMMETRY OPERATIONS ON THE BASIS VECTORS OF FIG. 3.10(a)

	E	C_3	C_3^2	σ_1	σ_2	σ_3
e_1	e_1	e_3	e_5	e_1	e_5	e_3
e_2	e_2	e_4	e_6	$-e_2$	$-e_6$	$-e_4$
e_3	e_3	e_5	e_1	e_5	e_3	e_1
e_4	e_4	e_6	e_2	$-e_6$	$-e_4$	$-e_2$
e_5	e_5	e_1	e_3	e_3	e_1	e_5
e_6	e_6	e_2	e_4	$-e_4$	$-e_2$	$-e_6$

Therefore the 6-dimensional representation matrices are,

$$D(E) = \begin{bmatrix} I & & \\ & I & \\ & & I \end{bmatrix} \quad D(C_3) = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ I & 0 & 0 \end{bmatrix} \quad D(C_3^2) = \begin{bmatrix} 0 & 0 & I \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix}$$

where I = 2 x 2 identity matrix. These matrices are special cases of the set of matrices A_1, A_2 .... A_m of section 3.2.

$$D(\sigma_1) = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & 0 & I_1 \\ 0 & I_1 & 0 \end{bmatrix}, \quad (D(\sigma_2) = \begin{bmatrix} 0 & 0 & I_1 \\ 0 & I_1 & 0 \\ I_1 & 0 & 0 \end{bmatrix},$$

$$D(\sigma_3) = \begin{bmatrix} 0 & I_1 & 0 \\ I_1 & 0 & 0 \\ 0 & 0 & I_1 \end{bmatrix} \quad \text{where } I_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The characters for this representation are,

$$k(E) = 6, \quad k(C_3, C_3^2) = 0, \; k(\sigma_1, \sigma_2, \sigma_3) = 0$$

Therefore if the above representation is **reduced** to $R = c_1R_1 \oplus c_2R_2 \oplus c_3 R_3$ where R_1, R_2 and R_3 are available from Table 4.9 , and from eq. (10) of the IV Chapter and the Table 4.10, one has,

$$c_1 \frac{6 \times 1}{6} = c_2 = 1 \quad \text{and} \quad c_3 \frac{6 \times 2}{6} = 2$$

Thus $R = R_1 \oplus R_2 \oplus 2R_3$ which implies that the stiffnes matrix in symmetry adapted basis will break to one block of 1 x 1, one block 1 x 1 and two identical blocks of 2 x2.

The projection operators

$$P_{ij}^{(k)} = \sum_{l=1}^{n} D_{kij}(g_l)\, g_l, \quad \text{for this case are}$$

following:

$$P^{(1)}_{11} = E + C_3 + C_3^2 + \sigma_1 + \sigma_2 + \sigma_3$$

$$P^{(2)}_{11} = E + C_3 + C_3^2 - \sigma_1 - \sigma_2 - \sigma_3$$

$$P^{(3)}_{11} = E - \tfrac{1}{2} C_3 - \tfrac{1}{2} C_3^2 - \sigma_1 + \tfrac{1}{2}\sigma_2 + \tfrac{1}{2}\sigma_3 \qquad (8)$$

$$P^{(3)}_{22} = E - \tfrac{1}{2} C_3 - \tfrac{1}{2} C_3^2 + \sigma_1 - \tfrac{1}{2}\sigma_2 - \tfrac{1}{2}\sigma_3$$

$$P^{(1)}_{12} = \frac{\sqrt{3}}{2}\left(-C_3 + C_3^2 + \sigma_2 - \sigma_3\right)$$

$$P^{(1)}_{21} = \frac{\sqrt{3}}{2}\left(C_3 - C_3^2 + \sigma_2 - \sigma_3\right)$$

The symmetry adapted basis vectors are any 6 linearly independent normalised vectors out of $P^{(k)}_{ij}\, e_1$. By using Table 5.1 and the first four projection operators one has the symmetry adapted basis vectors.

$$\bar{e}_1 = \frac{1}{\sqrt{3}}(e_1 + e_3 + e_5), \qquad \bar{e}_2 = \frac{1}{\sqrt{3}}(e_2 + e_4 + e_6)$$

$$\bar{e}_3 = \frac{1}{\sqrt{6}}(2e_1 - e_3 - e_5), \qquad \bar{e}_4 = \frac{1}{\sqrt{6}}(2e_2 - e_4 - e_6)$$

$$\bar{e}_5 = \frac{1}{\sqrt{2}}(e_3 - e_5), \qquad \bar{e}_6 = \frac{1}{\sqrt{2}}(e_4 - e_6)$$

Therefore if

$$\begin{bmatrix} \bar{e}_1 \\ \bar{e}_2 \\ \vdots \\ \bar{e}_6 \end{bmatrix} = [S] \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_6 \end{bmatrix}$$

Then

$$S = \left[\begin{array}{cc|cc|cc}
\frac{1}{\sqrt{3}} & & \frac{1}{\sqrt{3}} & & \frac{1}{\sqrt{3}} & \\
 & \frac{1}{\sqrt{3}} & & \frac{1}{\sqrt{3}} & & \frac{1}{\sqrt{3}} \\
\hline
\sqrt{\frac{2}{3}} & & -\frac{1}{\sqrt{6}} & & -\frac{1}{\sqrt{6}} & \\
 & \sqrt{\frac{2}{3}} & & -\frac{1}{\sqrt{6}} & & -\frac{1}{\sqrt{6}} \\
\hline
 & & \frac{1}{\sqrt{2}} & & -\frac{1}{\sqrt{2}} & \\
 & & & \frac{1}{\sqrt{2}} & & -\frac{1}{\sqrt{2}}
\end{array}\right]$$

Therefore $\bar{K} = SKS^{-1} = SKS^{T}$ is the stiffness matrix in the symmetry adapted basis. Putting the expression for K from III Chapter (Example - 2 of section 3.3) one has,

$$K = \frac{1}{\sqrt{3}} \begin{bmatrix} I & I & I \\ \sqrt{2}I & -\frac{I}{\sqrt{2}} & -\frac{I}{\sqrt{2}} \\ 0 & \sqrt{\frac{3}{2}}I & -\sqrt{\frac{3}{2}}\,I \end{bmatrix} \begin{bmatrix} A & B & B^{T} \\ B^{T} & A & B \\ B & B^{T} & A \end{bmatrix} \begin{bmatrix} I & \sqrt{2}I & 0 \\ I & -\frac{I}{\sqrt{2}} & \sqrt{\frac{3}{2}}I \\ I & -\frac{I}{\sqrt{2}} & -\sqrt{\frac{3}{2}}I \end{bmatrix}$$

$$= \begin{bmatrix} A+B+B^{T} & 0 & 0 \\ 0 & A - \frac{1}{2}(B+B^{T}) & -\frac{3}{2}\,(B^{T}-B) \\ 0 & \frac{3}{2}(B^{T}-B) & A-\frac{1}{2}(B+B^{T}) \end{bmatrix}$$

Therefore,

$$\overline{K} = \left[\begin{array}{cc|cc|cc}
2k_2 & 0 & & & & \\
0 & k_3+3k_1 & & & & \\
\hline
& & 2k_2 + \frac{3k_1}{4} & 0 & 0 & \frac{3}{4}k_1 \\
& & 0 & k_3 & -\frac{3}{4}k_1 & 0 \\
\hline
& & 0 & -\frac{3}{4}k_1 & 2k_2+\frac{3k_1}{4} & 0 \\
& & \frac{3}{4}k_1 & 0 & 0 & k_3
\end{array}\right]$$

where,

Corresponding to $\frac{1}{\sqrt{3}}\ (e_1 + e_3 + e_5)$, $\overline{K}$ has its Ist row

" " $\frac{1}{\sqrt{3}}\ (e_2 + e_4 + e_6)$, $\overline{K}$ has its IInd row

Corresponding to $\frac{1}{\sqrt{6}}\ (2e_1 - e_3 - e_5)$, $\overline{K}$ has its IIIrd row

Corresponding to $\frac{1}{\sqrt{6}}(2e_2 - e_4 - e_6)$, $\overline{K}$ has its IVth. row

Corresponding to $\frac{1}{\sqrt{2}}\ (e_3 - e_5)$, $\overline{K}$ has its Vth row

Corresponding to $\frac{1}{\sqrt{2}}\ (e_4 - e_6)$, $\overline{K}$ has its VIth row

When these basis vectors are arranged one has,

$$\overline{K} = \left[\begin{array}{cc|cc|cc}
2k_2 & & & & & \\
& k_3+3k_1 & & & & \\
\hline
& & 2k_2+\frac{3}{4}k_1 & \frac{3}{4}k_1 & & \\
& & \frac{3}{4}k_1 & k_3+\frac{3}{4}k_1 & & \\
\hline
& & & & 2k_2+\frac{3}{4}k_1 & \frac{3}{4}k_1 \\
& & & & \frac{3}{4}k_1 & k_3+\frac{3}{4}k_1
\end{array}\right]$$

where,

Corresponding to $\frac{1}{\sqrt{3}}$ $(e_1 + e_3 + e_5)$, this $\bar{K}$ has its Ist row

Corresponding to $\frac{1}{\sqrt{3}}$ $(e_2 + e_4 + e_6)$, " " " IInd "

Corresponding to $\frac{1}{\sqrt{6}}$ $(2e_1 - e_3 - e_5)$, " " " IIIrd "

Corresponding to $\frac{1}{\sqrt{2}}$ $(2e_1 - e_3 - e_5)$, " " " IVthe "

Corresponding to $\frac{1}{\sqrt{6}}$ $(2e_2 - e_4 - e_6)$, " " " Vth "

Corresponding to $\frac{1}{\sqrt{2}}$ $(e_5 - e_3)$, " " " VIth row.

Therefore two frequencies are doubly degenerate because the same block is occurring twice. The frequencis are as were found in IIIrd Chapter. The mode shapes are $\frac{1}{\sqrt{3}}(e_1 + e_3 + e_5)$; $\frac{1}{\sqrt{3}}(e_2 + e_4 + e_6)$, $\frac{x_1}{\sqrt{6}}(2e_1 - e_3 - e_5) + \frac{x_2}{\sqrt{2}}(e_4 - e_6)$, $\frac{x_1}{\sqrt{6}}(2e_2 - e_4 - e_6) + \frac{x_2}{\sqrt{2}}(e_5 - e_3)$ where x_1 and x_2 will have two values one corresponding to each frequency. Thus written in vector form, mode shapes are

$$\frac{1}{\sqrt{3}}\begin{bmatrix}1\\0\\1\\0\\1\\0\end{bmatrix}, \quad \frac{1}{\sqrt{3}}\begin{bmatrix}0\\1\\0\\1\\0\\1\end{bmatrix}, \quad \frac{1}{\sqrt{2}}\begin{bmatrix}2x_1/\sqrt{3}\\0\\-x_1/\sqrt{3}\\x_2\\-x_1/\sqrt{3}\\-x_2\end{bmatrix}, \quad \frac{1}{\sqrt{2}}\begin{bmatrix}0\\2x_1/\ 3\\-x_2\\-x_1/\sqrt{3}\\x_2\\-x_1/\sqrt{3}\end{bmatrix}.$$

where x_1 and x_2 have two values giving altogether 6 modes. Also $x_1^2 + x_2^2 = 1$ and $\frac{x_1}{x_2}$ depends upon the relative magnitudes of k_1, k_2 and k_3.

5.4.2 <u>The Group C_{4v}</u>

A few of the system whose symmetry group is C_{4v}, are shown in Fig. 3.9 and 5.2. For the system shown in Fig.(3.9) there are 16 degrees of freedom. The basis vectors are so chosen (as in example 1 of section 3.3) that effect of various symmetry operations are as shown in Table 5.2.

TABLE 5.2

EFFECTS OF SYMMETRY OPERATIONS ON BASIS VECTORS OF FIG. 3.9

	E	C_4	C_4^2	C_4^3	σ_1	σ_2	σ_3	σ_4
e_1	e_1	e_5	e_9	e_{13}	e_3	e_{15}	e_{11}	e_7
e_2	e_2	e_6	e_{10}	e_{14}	e_4	e_{16}	e_{12}	e_8
e_3	e_3	e_7	e_{11}	e_{15}	e_1	e_{13}	e_9	e_5
e_4	e_4	e_8	e_{12}	e_{16}	e_2	e_{14}	e_{10}	e_6
e_5	e_5	e_9	e_{13}	e_1	e_{15}	e_{11}	e_7	e_3
e_6	e_6	e_{10}	e_{14}	e_2	e_{16}	e_{12}	e_8	e_4
e_7	e_7	e_{11}	e_{15}	e_3	e_{13}	e_9	e_5	e_1
e_8	e_8	e_{12}	e_{16}	e_4	e_{14}	e_{10}	e_6	e_2
e_9	e_9	e_{13}	e_1	e_5	e_{11}	e_7	e_3	e_{15}
e_{10}	e_{10}	e_{14}	e_2	e_6	e_{12}	e_8	e_4	e_{16}
e_{11}	e_{11}	e_{15}	e_3	e_7	e_9	e_5	e_1	e_{13}
e_{12}	e_{12}	e_{16}	e_4	e_8	e_{10}	e_6	e_2	e_{14}
e_{13}	e_{13}	e_1	e_5	e_9	e_7	e_3	e_{15}	e_{11}
e_{14}	e_{14}	e_2	e_6	e_{10}	e_8	e_4	e_{16}	e_{12}
e_{15}	e_{15}	e_3	e_7	e_{11}	e_5	e_1	e_{13}	e_9
e_{16}	e_{16}	e_4	e_8	e_{12}	e_6	e_2	e_{14}	e_{10}

The 16-dimensional representation of C_{4v} in this basis is given by ,

$$D(E) = \begin{bmatrix} I & & & \\ & I & & \\ & & I & \\ & & & I \end{bmatrix}, \quad D(C_4) = \begin{bmatrix} & I & & \\ & & I & \\ & & & I \\ I & & & \end{bmatrix}, \quad D(C_4^2) = \begin{bmatrix} & & I & \\ & & & I \\ I & & & \\ & I & & \end{bmatrix}$$

$$D(C_4^3) = \begin{bmatrix} & & & I \\ I & & & \\ & I & & \\ & & I & \end{bmatrix}$$ where I is identity matrix of 4 x 4.

and

$$D(\sigma_1) = \begin{bmatrix} & I & & & & & & \\ I & & & & & & & \\ & & & & & & & I \\ & & & & & & I & \\ & & & & & I & & \\ & & & & I & & & \\ & & & I & & & & \\ & & I & & & & & \end{bmatrix} \qquad D(\sigma_2) = \begin{bmatrix} & & & & & & & I \\ & & & & & & I & \\ & & & & & I & & \\ & & & & I & & & \\ & & & I & & & & \\ & & I & & & & & \\ & I & & & & & & \\ I & & & & & & & \end{bmatrix},$$

$$D(\sigma_3) = \begin{bmatrix} & & & I & & \\ & & I & & & \\ & I & & & & \\ I & & & & & \\ & & & & & I \\ & & & & I & \end{bmatrix} \qquad D(\sigma_4) = \begin{bmatrix} & & & I & & & & \\ & & I & & & & & \\ & I & & & & & & \\ I & & & & & & & \\ & & & & & & & I \\ & & & & & & I & \\ & & & & & I & & \\ & & & & I & & & \end{bmatrix}$$

Where I

identity

As discussed earlier, this representation reduces to $R = 2R_1 \oplus 2R_2 \oplus 2R_3 \oplus 2R_4 \oplus 4R_5$ in the symmetry adapted basis which implies that the stiffness matrix for this problem will break to four 2 x 2 blocks and 2 identical 4x4 blocks which means one has to invert only four 2 x 2 matrices and one 4 x4 matrix in analysis and in buckling or vibration problem one has to solve four 2 x 2 and one 4 x 4 determinant.

The projection operators $P_{ij}^{(k)} = \sum_{l=1}^{n} D_{kij}(g_l)\, g_l$,

which generate the symmetry adapted basis are,

$$
\begin{aligned}
P_{11}^{(1)} &= E + C_4 + C_4^2 + C_4^3 + \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 \\
P_{11}^{(2)} &= E + C_4 + C_4^2 + C_4^3 - \sigma_1 - \sigma_2 - \sigma_3 - \sigma_4 \\
P_{11}^{(3)} &= E - C_4 + C_4^2 - C_4^3 + \sigma_1 - \sigma_2 + \sigma_3 - \sigma_4 \\
P_{11}^{(4)} &= E - C_4 + C_4^2 - C_4^3 - \sigma_1 + \sigma_2 - \sigma_3 + \sigma_4 \\
P_{11}^{(5)} &= E - C_4^2 + \sigma_1 - \sigma_3 \\
P_{22}^{(5)} &= E - C_4^2 - \sigma_1 + \sigma_3 \\
P_{12}^{(5)} &= - C_4 + C_4^3 - \sigma_2 + \sigma_4 \\
P_{21}^{(6)} &= C_4 - C_4^3 - \sigma_2 + \sigma_4
\end{aligned}
\qquad ..(9)
$$

Eq. (9) will be of use in next section. Following the routine of section 5.4.1 one can get the symmetry adapted basis for this case also and hence the transformation matrix S which can be used to block diagonalise the stiffness matrix of example -1 of section 3.3 to four 2 x 2 and two identical 4 x 4 blocks (the basis vectors should be arranged according to the symmetry species of R^5 after the transformation has been carried out. In fact this is a must in every problem with non-Abelian symmetry group

in which case one has at least one irreducible representation as two dimensional.

All the above results are applicable to cable system shown in Fig. 3.8 if it is symmetrised from C_{2v} to C_{4v}. Thus the question posed in discussion of example-2 of the section 3.2 is answered. The resulting simplification is that here one has to solve four 2 x 2 determinants and one 4 x 4 determinant instead of four 4 x 4 determinants as was the case in that example.

Similar analysis can be applied to systems shown in Figs. 2.14 and 2.15 which have got symmetry groups C_{6v} whose irreducible representations are shown in Appendix Table A-5. A corresponding analysis for any general C_{nv} is reserved for later works.

5.5 SYMMETRIES OF NORMAL MODES IN NON-LINEAR VIBRATIONS:

A system linear or nonlinear is said to vibrate in normal mode when, ((56) and (57)),

(i) motion is periodic (i.e. all the degrees of freedom have same period).

(ii) all the degrees of freedom pass through their equilibrium positions at the same time

(iii) the displacements in all the degrees of freedom can be uniquely determined in terms of the displacement in one of the degrees of freedom.

However, in nonlinear systems the number of mode shapes can be more than the number of degrees of freedom (56) and (57). This distinction possibly arises due to the phrase "uniquely determined" in (iii), which is linear relationship in linear systems and may be any thing in non-linear systems (including the linear relationship in which case the non-linear system is said to vibrate in "similar normal mode" (56, 57). In the following only similar normal modes are considered. The details of the work are not given and only the results of Yang (31) are sketched with their justifications.

Let the system has N-degrees of freedom and let $U(X)$ be the deformation potential energy then,

1. The Yang's $N \times N$ isometries $A, B, C, \ldots$, (where $A, B, C \ldots$ are such that $U(AX) = U(X)$, $U(BX) = U(X)$ etc.) which he finds by inspection, are nothing but the N-dimensional representation matrices of the symmetry group which the system has.

The truth of this follows from eq. 5.1, i.e. $U(RX) = U(X)$ where $R = \{D(g_1), D(g_2) \ldots\ldots D(g_n)\}$ is the N-dimensional representation of the symmetry group $G = \{g_1, g_2 \ldots\ldots g_n\}$ associated with the system.

Thus one has 'n' $N \times N$ matrices under which the energy $U(X)$ remains invariant. Therefore in the IInd example of Yang (31) there will be 8 matrices of 8×8

because the symmetry group is of order 8. These matrices will be the 8-dimensional representation matrices of C_{4v} and can be found by a definite method and not just by inspection. By inspection one can not know as to how many isometries will be there in a given problem.

(2) The invariant subspaces of N-dimensional phase space (an invariant subspace is that subspace in which X and grad U(X) both lie) which Yang finds by finding the eigenvectors of the isometries A, B, C .... AB or ABA etc., are generated by the projection operators of the present work.

The truth of this follows from comparison of the definitions of the projection operators and the invariant subspaces.

EXAMPLE 1

Consider the system shown in Fig. 5.1 where the springs are non-linear. The potential energy of deformation is, $U(X) = F_1(x_1) + F_1(x_2) + F_2(x_1 + x_2)$

The symmetry group is $C_{1h} = \{E, \sigma\}$ whose 2-dimensional representation is $R = \{D(E), D(\sigma)\}$, where $D(E) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $D(\sigma) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ in the above basis, and $U(RX) = U(X)$.

Note that $D(\sigma)$ would have been Yang's isometry for this system. The projection operators for the group C_{1h} are $P^{(1)} = E + \sigma$ and $P^{(2)} = E - \sigma$ and then $P^{(1)}x_1 = x_1 + x_2$,

$P^{(2)}x_1 = x_1 - x_2$, ($P^{(1)}x_2$, $P^{(2)}$ x_2 would give nothing different from the $P^{(1)}x_1$, $P^{(2)}x_1$). Thus $x_1 + x_2 = 0$ and $x_1 - x_2 = 0$ are Yang's invariant sub-spaces of U(X) which define two straight lines orthogonal to each other. These are the two modes of the system.

EXAMPLE - 2

Consider the system shown in Fig. 5.2 which is same as that of Yang's IInd example and for the sake of clear comparisions the same notations are used.

The initial basis is chosen as shown in Fig. 5.2 (which is very inconvenient basis however, but for the sake of comparision it has been adopted). The system has the symmetry group $C_{4v} = \{E, C_4, C_4^2, C_4^3, \sigma_1, \sigma_2, \sigma_3, \sigma_4\}$. There are 8-degrees of freedom. The effects of symmetry operations on these basis vectors is shown in Table 5.3.

TABLE 5.3

THE EFFECTS OF SYMMETRY OPERATIONS ON THE BASIS VECTORS OF FIG. 5.2

	E	C_4	C_4^2	C_4^3	σ_1	σ_2	σ_3	σ_4
u_1	u_1	v_2	$-u_3$	$-v_4$	v_1	u_4	$-v_3$	$-u_2$
u_2	u_2	v_3	$-u_4$	$-v_1$	v_4	u_3	$-v_2$	$-u_1$
u_3	u_3	v_4	$-u_1$	$-v_2$	v_3	u_2	$-v_1$	$-u_4$
u_4	u_4	v_1	$-u_2$	$-v_3$	v_2	u_1	$-v_4$	$-u_3$
v_1	v_1	$-u_2$	$-v_3$	u_4	u_1	$-v_4$	$-u_3$	v_2
v_2	v_2	$-u_3$	$-v_4$	u_1	u_4	$-v_3$	$-u_2$	v_1
v_3	v_3	$-u_4$	$-v_1$	u_2	u_3	$-v_2$	$-u_1$	v_4
v_4	v_4	$-u_1$	$-v_2$	u_3	u_2	$-v_1$	$-u_4$	v_3

Then the 8-dimensional representation R is given by the matrices, $D(E) = 8 \times 8$ identity matrix and

$$D(C_4) = \begin{bmatrix} 0 & c \\ -c & 0 \end{bmatrix}, \quad D(C_4^2) = \begin{bmatrix} -d & 0 \\ 0 & -d \end{bmatrix} \quad D(C_4^3) = \begin{bmatrix} 0 & -c \\ c & 0 \end{bmatrix}$$

$$D(\sigma_1) = \begin{bmatrix} 0 & bc \\ bc & 0 \end{bmatrix} \quad D(\sigma_2) = \begin{bmatrix} b & 0 \\ 0 & -b \end{bmatrix} \quad D(\sigma_3) = \begin{bmatrix} 0 & -ac \\ -ac & 0 \end{bmatrix}$$

and $$D(\sigma_4) = \begin{bmatrix} -a & 0 \\ 0 & a \end{bmatrix}$$

where
$$a = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad b = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad c = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

and $$d = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

Note that the present $D(\sigma_4)$, $D(\sigma_1)$ and $D(C_4^2)$ are the Yang's isometries A, C and D respectively. Under all the above 8-matrices the deformation potential energy $U(X)$, where $X = (u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4)$, remains invariant as can be verified from,

$$U(X) = F(((1 + u_2 - u_1)^2 + (v_2 - v_1)^2)^{\frac{1}{2}} - 1)$$

$$+ F(((1 + u_3 - u_4)^2 + (v_3 - v_4)^2)^{\frac{1}{2}} - 1)$$

$$+ F(((1 + v_4 - v_1)^2 + (u_4 - u_1)^2)^{\frac{1}{2}} - l)$$

$$+ F(((1 + v_3 - v_2)^2 + (u_3 - u_2)^2)^{\frac{1}{2}} - l)$$

where l = the length of the springs and it has been assumed that potential energy results only due to change in length.

Now there are 8 isometries (instead of Yang's 4 isometries) which have been obtained by a definite procedure. One can now use the Yangs procedure with these 8 matrices to get all that is required.

However the above procedure is not needed and even the isometries are not needed. Only the knowledge of the Table 5.3 and the eq. (9) is required and it can be shown that all the invariant subspaces are obtained by $P_{ij}^{(k)} \cdot (u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4)$ out of which only 8 will be linearly independent.

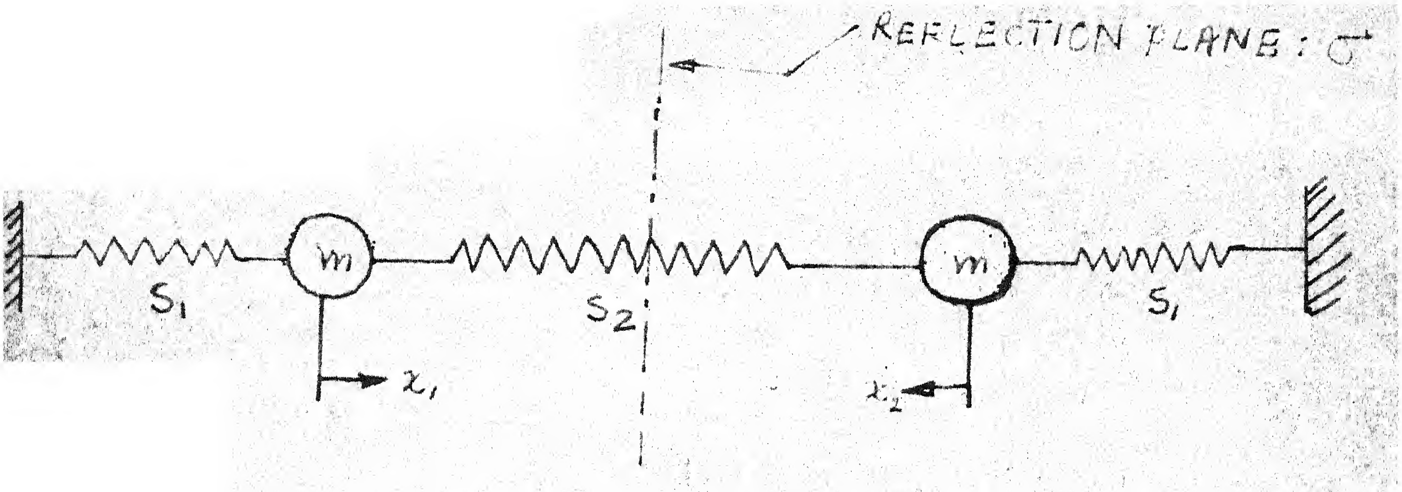


FIG. 5.1
NONLINEAR SPRING–MASS SYSTEM WITH THE SYMMETRY GROUP C_{1h}.

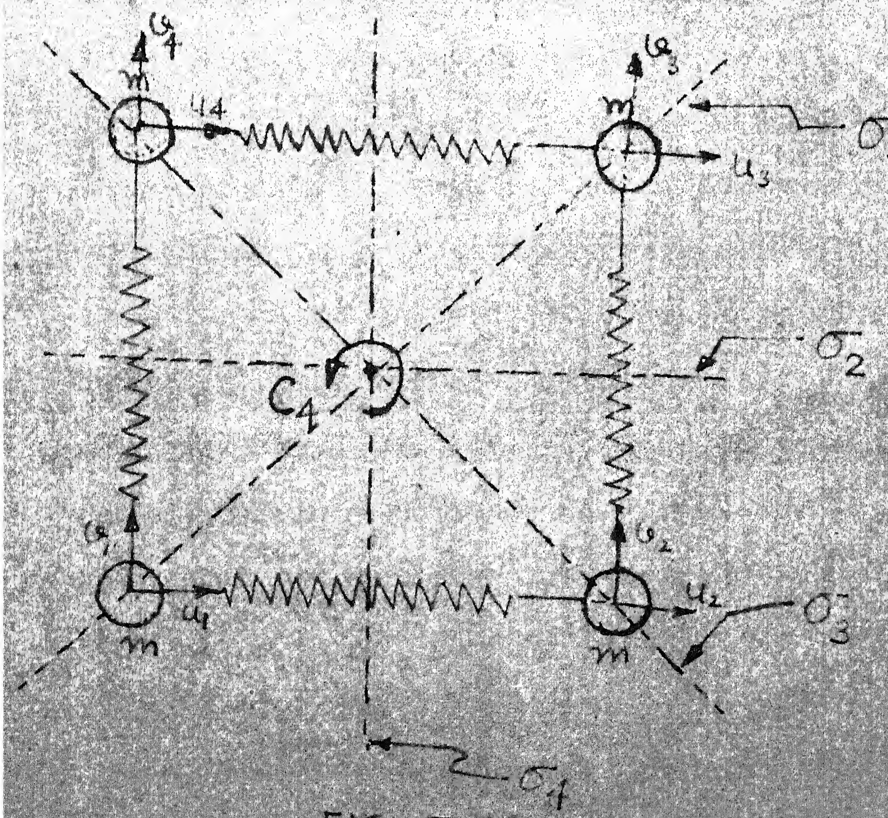


FIG. 5.2
NONLINEAR SPRING–MASS SYSTEM WITH THE SYMMETRY GROUP C_{4v}.

CHAPTER 6

CONCLUSIONS AND RECOMMENDATIONS FOR FURTHER RESEARCH

The vague word symmetry with regard to structural systems has been precisely and elegantly defined with the help of what has been called the symmetry elements and symmetry operations. It has been shown that there is a direct relation between the structural symmetries and the symmetries of their stiffness matrices provided suitable co-ordinate systems are chosen and proper numbering of the degrees of freedom is done, e.g. the mirror symmetric structural systems have centro-symmetric stiffness matrices, the cyclically symmetric structural systems have cyclically symmetric stiffness matrices etc. These symmetries of the stiffness matrices follow from their invariance under similarity transformations by a set of matrices which represent the symmetry transformations of the structural systems. It is also seen that the structural symmetries have nothing to do with the load symmetries except in the buckling problems where the stiffness matrices are functions of axial loads. The structural systems can also be regarded symmetric in the sense of the symmetries of their stiffness matrices even without

any trace of symmetry in geometry, topology or member aggregate of the systems. However, it has been assumed that geometry, topology and member aggregate contribute equally to the symmetry of structural systems.

The problems of reflection symmetric, double reflection symmetric cyclically symmetric (and their combination) structural systems have been simplified by similiarity transformation of stiffness matrices by the matrices which are formed from eigen vectors of some sets of simple matrices which commute with the stiffness matrices and with themselves also and are nothing but the symmetry transformation matrices of the system and are easily obtained. Under such similiarity transformation the stiffness matrices get block diagonalised. At this stage a famous theorem in matrix theory and very much used in Quantum Mechanics convinces about the above similarity transformation. However, some inherent difficulties are encountered in such a procedure e.g. (i) to be able to simplify the problem a particular choice of the co-ordinate systems and a particular numbering of the degrees of freedom is required (ii) the problems with some nodes comming on the axis of symmetry, on the planes of symmetry can not be dealt and thus the full symmetry of a structural system may not be used. (iii) the effects of increasing or decreasing the symmetries (i.e. the symmetry elements) can not be seen from the out set etc.

Due to these inadequacies and difficulties in such a classical procedure, a need for better procedure is thought. This better procedure is to handle the symmetries through algebra only and is accomplished by the 'group theory', the mathematics of symmetry.

It is seen that the symmetry elements of any structural system form a group which is called the symmetry group of the system. Thus hundreds and thousands of possible structural systems can be classified according to their symmetry groups and therefore dealing with all possible symmetric structural systems boils down to the study of a few symmetry groups. The representation theory and the character tables of symmetry groups turned out to be very useful to tell the block diagonal forms of stiffness matrices obtainable through symmetry only. One can get the quantitative information about the possible simplifications due to symmetry, even without actually determining the stiffness matrices. One can also know the effects of increasing or decreasing the symmetry elements of a structural system just from the out set by knowing the resulting symmetry groups due to increase or decrease of the symmetry elements.

The co-ordinate system in which the stiffness matrices get block diagonalised are generated by what has been called the projection operators, when applied to the basis

vectors of initial co-ordinate systems. For every group there is a unique set of these operators obtainable without any labour, from the irreducible representations and symmetry elements of the group. This co-ordinate system is called the symmetry adapted basis for the structural system.

The application of group theory is in no way restricted by nodes on the symmetry planes or symmetry axes. The initial choice of nodal co-ordinate systems and numbering of degrees of freedom can be arbitrary. However a judicious choice of initial co-ordinate systems and the proper numbering of the degrees of freedom reduces the intermediate labour by quite a lot. Since many of the symmetry groups have similar algebraic nature, therefore the symmetries corresponding to those many groups can be treated at a stretch which is not the case with the classical methods. Also the symmetry groups can be used either in full or in part (subgroups) to simplify the problems.

The procedure developed in the present work yield the results of earlier authors (20, 21, 22, 23) as special cases.

Following is the procedure to apply group theory.

(1) Recognise a few symmetry elements of the structural system.

(2) Find the symmetry group.

(3) Find the irreducible representations and character table for the group or use the ready made tables if available.

(4) Find the projection operators of the symmetry group.

(5) Choose a suitable initial basis and properly number the various degrees of freedom so as to get simplest forms of representation matrices.

(6) Find the effects of various symmetry operations on the basis vectors.

(7) Find the representation of the group whose dimension is equal to the number of degrees of freedom. Find the characters also.

(8) Find as to how many time each irreducible representation is contained in this representation. These numbers give the dimensions of the blocks in which stiffness matrix will break through the use of symmetry only.

(9) Apply the projection operators to the initial basis vectors and normalise the resulting linear combinations of the initial vectors. This will give the set of linearly independent vectors. These vectors will be, what has been called the symmetry adapted basis vectors.

(10) Find the transformation matrix, whose rows are the coefficients of the initial basis vectors in the expressions for symmetry adapted basis vectors.

(11) Apply transformation by the above matrix.

(12) Arrange the rows and columns of the transformed stiffness matrix according to symmetry species of the irreducible representations. This will give the block diagonal form to the stiffness matrix.

By using the methods developed in the III Chapter a few problems have been solved by hand. Had the present method not been used, it would have been rather impossible to solve them by hand (e.g. solving 16 x 16 determinantal equation or solving 6 x 6 determinantal equation of a fully populated matrix or solving 16 simultaneous linear algebraic equations etc.) Thus the procedure of III Chapter is quite helpful. Group theory, however, when applied to the system shown in Fig. 3.9 or 3.10(a) reduces the problems to more simple forms. The results of Yang (31) have also followed by use of group theory in a very straight forward manner.

Quite a number of problems are there for further investigations. Application of group theory to structural systems with symmetry group C_{nv} (for all n) is one of the work left for future. Application of group theory to non-linear structural systems is also left for further work.

The chaotic . equations occurring in theory of shells may be related to one another by some group of transformations. The application of group theory may be very fruitful in finite element methods where there are thousands of degrees of freedom.

APPENDIX

THE IRREDUCIBLE REPRESENTATIONS AND CHARACTER TABLES OF VARIOUS SYMMETRY GROUPS

The set of tables given below are the irreducible representations of various groups. These tables are already available in the treatises on group theory. These are quoted here, for reference. At every stage, if one wants to use group theory for simplifying the problems of symmetrical structural systems, these tables are required. They can be derived by using the materials available in IV Chapter. Note that several groups have same irreducible representation. e.g. C_{1h}, C_2 and S_2 etc. all have same tables. Also note that the groups C_{nh}, C_n, C_{2v}, etc. are all Abelian groups and therefore for them the character tables and irreducible representations are one and the same and therefore no seprate table for character is given. R_1, R_2 .... R_r stand for the irreducible representations.

A-1 Groups With All One Dimensional Irreducible Representations

Table A-1

The Groups C_2, C_{1h} and S_2

C_2	E	C_2
C_{1h}	E	
S_2	E	i
R_1	1	1
R_2	1	-1

Table A-2

The Groups C_{2v}, C_{2h} and D_2

C_{2v}	E	C_2	1	2
C_{2h}	E	C_2	i	h
D_2	E	C_2	C_2^a	C_2^b
R_1	1	1	1	1
R_2	1	1	-1	-1
R_3	1	-1	1	-1
R_4	1	-1	-1	1

Table A-3

The Groups C_4 and S_4

C_4	E	C_4^2	C_4	C_4^3
S_4	E	C_4^2	S_4	S_4^3
R_1	1	1	1	1
R_2	1	1	-1	-1
R_3	1	-1	i	-i
R_4	1	-1	-i	i

Table A-4

The Groups C_6, C_{3h}, S_6

C_6	E	C_6^2	C_6^4	C_6^3	C_6^2	C_6
C_{3h}	E	C_3	C_3^2	h	S_3	S_3^2
R_1	1	1	1	1	1	1
R_2	1	1	1	-1	-1	-1
R_3	1	$-\theta^*$	$-\theta$	-1	θ^*	θ
R_4	1	$-\theta$	$-\theta^*$	-1	θ	θ^*
R_5	1	$-\theta^*$	$-\theta$	1	$-\theta^*$	$-\theta^*$
R_6	1	$-\theta$	$-\theta^*$	1	$-\theta$	$-\theta^*$

where $\theta = e^{\frac{2\pi i}{6}}$

The table for C_n is given in IV Chapter (Table 4.12), the table for D_{2h} can be constructed with the help of tables A-1 and A-2 by observing that D_{2h} is direct product of S_2 and D_2.

A-2 Groups Which Have Some of the Irreducible Representations 2-Dimensional.

The table for group C_{3v} and C_{4v} are given in Tables 4.9 and 4.11 (a). Their character table is shown Tables 4.10 and 4.11(b). The tables for the groups D_3 and D_4, and D_{2d} are same as that for C_{3v} and C_{4v}.

TABLE A-5

The Group C_{6v}

		1	2	3	4	5	6	For top row multiply entries by
		E	C_6^2	C_6^4	C_6^3	C_6	C_6^5	
R_1		1	1	1	1	1	1	1
R_2		1	1	1	1	1	1	−1
R_3		1	1	1	−1	−1	−1	1
R_4		1	1	1	−1	−1	−1	−1
	11	1	−c	−c	−1	c	c	1
R_5	12	0	−s	s	0	−s	s	1
	21	0	s	−s	0	s	−s	−1
	22	1	−c	−c	−1	c	c	−1
	11	1	−c	−c	1	−c	−c	1
R_6	12	0	−s	s	0	s	−s	1
	21	0	s	−s	0	−s	s	−1
	22	1	−c	−c	1	−c	−c	−1

where $c = \cos 60°$, $s = \sin 60°$ and ij $(i, j = 1,2)$ right to R_5 or R_6 means the ij element of the R_5 or R_6 matrices.

Tables for D_{3h}, D_{3d}, D_6, C_{6h}, D_{6h}, C_{4h} and D_{4h} etc. follow from these tables by observing that these groups are direct products of some two groups for which tables are given here.

The characters for C_{6v} for R_1, R_2, R_3 and R_4 are same as representations (because these representations

are one-dimensional). The characters for R_5 and R_6 are given below:

	E	(C_6^2, C_6^4)	(C_6, C_6^5)	C_6^3	$(\sigma_1, \sigma_2, \cdots \sigma_6)$
R_5	2	$-2c$	$2c$	-2	0
R_6	2	$-2c$	$-2c$	2	0

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